Analytic Properties of Radial Wave Functions*

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This is a review article about the properties of radial wave functions and other quantities relevant to the partial wave analysis of scattering theory, as functions of the energy or wave number. The treatment is restricted to the nonrelativistic Schrödinger equation for two particles with a local potential. In addition to regular and irregular solutions of the radial differential equations, the Jost function, S matrix, and Green's functions are analyzed and completeness is proved. The examples investigated in detail include the Bargmann potentials and their generalizations.

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1. INTRODUCTION

QUANTUM mechanics has undoubtedly its most beautiful general form in the language of abstract vector space theory, whose mathematical methods furnish it at the same time with one of its most powerful tools. There are nevertheless many fundamental problems which are attacked with advantage in a special representation. It has become clear lately that in nonrelativistic quantum mechanics as well as in relativistic field theory much can be gained by returning from the formal operator calculus to that of point functions. The theory of functions of complex variables, specifically, has become again a prominent tool of physics.

The recent upsurge of dispersion relation research is a case in point. Although its results can frequently be obtained without ever going into the complex plane, the most appropriate general mathematical tool is the theory of analytic functions.

Most physicists are quite conversant with the theory of differential and integral equations; most know also the essentials in the theory of functions. The combination of these two disciplines, however, is much less familiar to many. A useful purpose may therefore be served by reviewing what is known by means of complex analysis in a certain area of scattering theory.

I shall restrict myself to the nonrelativistic quantum mechanics of two-particle systems, that is, the one-particle Schrödinger equation in the center of mass system. The properties of the solutions of such a partial differential equation not being nearly as well understood as those of solutions of ordinary differential equations, a partial wave analysis is made which leads to single or coupled radial equations. The regular and irregular solutions of these as well as all the functions constructed from them for the purpose of scattering theory are to be investigated.

I shall restrict myself to local potentials. Certain types of nonlocality, such as spin-orbit forces introduce no changes whatever. Others may introduce only inessential complications. In the latter case references to appropriate papers will be given. The general case of nonlocal forces, however, is far more difficult and little is known about it.

The purpose of this article is not only to collect results; it is also didactic. The proofs therefore form an essential part of its methodological aim. How many physicists have actually seen a completeness proof, except for some very special functions?

Very little in this paper is new. Almost everything in it can be found in the published literature, directly or by implication. In contrast to some authors on the subject I shall not make a weak assumption concerning the potential and then stick to it. From time to time the assumptions will be explicitly strengthened in order to see what can be said then. The weakest hypothesis, always to be kept, is that the first and second absolute moments of the potential are finite; stronger ones to be made at various points are that the potential has an exponential tail or that it vanishes identically beyond a certain point. Since the earlier papers by Jost, Levinson, and others had a special purpose their authors were not interested in doing that explicitly, although some of the general consequences of a finite range follow immediately from their work and were known to them. How much more can be said if the potential vanishes beyond a point has been demonstrated particularly by the work of Humblet and Regge.

2. PRELIMINARIES ON SCATTERING THEORY

We start from the Schrödinger equation for two particles in the center of mass coordinate system:

$$[-(\hbar^2/2\mu)\nabla^2+H_I(\mathbf{r})]\psi(\mathbf{r})=E\psi(\mathbf{r}), \tag{2.1}$$

μ being the reduced mass of the particles and $\mathbf{r}$, their

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relative distance. The interaction energy $H_I(\mathbf{r})$ is assumed to be invariant under rotations but may be spin dependent.

For the purpose of scattering theory it is advantageous to convert (2.1) into an integral equation which incorporates the boundary condition that at large distances the wave function should consist of a plane wave plus an outgoing spherical wave; thus, with $E=\hbar^2k^2/2\mu$,

$$\psi_+(\mathbf{k}s\nu,\mathbf{r})=\psi_0(\mathbf{k}s\nu,\mathbf{r})+\int(d\mathbf{r})G_+(\mathbf{k};\mathbf{r},\mathbf{r}')H_I(\mathbf{r}')\psi_+(\mathbf{k}s\nu,\mathbf{r}),\quad(2.2)$$

where

$$\psi_0(\mathbf{k}s\nu,\mathbf{r})=[(\mu k)^{\frac{1}{2}}/\hbar(2\pi)^{\frac{3}{2}}]\chi_s^{\nu}e^{i\mathbf{k}\cdot\mathbf{r}},$$

χ being the relevant normalized spin wave function for the intrinsic angular momentum of the two particles. The normalization of ψ_0 is such that

$$\int(d\mathbf{r})\psi_0^*(\mathbf{k}s\nu,\mathbf{r})\psi_0(\mathbf{k}'s'\nu',\mathbf{r})=\delta(E-E')\delta(\Omega_k-\Omega_k')\delta_{ss'}\delta_{\nu\nu'},$$

$$\sum_{s\nu}\int_0^\infty dE\int d\Omega_k\psi_0^*(\mathbf{k}s\nu,\mathbf{r})\psi_0(\mathbf{k}s\nu,\mathbf{r}')=\delta(\mathbf{r}-\mathbf{r}');$$

Ω_k is the solid angle defined by $\mathbf{k}$.

The specification of *outgoing* spherical waves is accomplished by the choice of Green's function:

$$G_+(k;\mathbf{r},\mathbf{r}')=\sum_{s\nu}\int d\Omega_k'\int_0^\infty dE\frac{\psi_0^*(\mathbf{k}'s\nu,\mathbf{r})\psi_0(\mathbf{k}'s\nu,\mathbf{r}')}{E-E'+i\epsilon}$$
$$=-\frac{\mu}{2\pi\hbar^2}\frac{\exp(ik|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|}$$
$$\underset{r\to\infty}{\sim}-\frac{e^{ikr}}{r}\left(\frac{2\pi\mu}{k\hbar^2}\right)^{\frac{1}{2}}\sum_{s\nu}\chi_s^{\nu}\psi_0^*(\mathbf{k}''s\nu,\mathbf{r}'),\quad(2.3)$$

where $\mathbf{k}''\equiv k\mathbf{r}r^{-1}$.

We expand the Green's function and wave functions in spherical harmonics:

$$G_+(k;\mathbf{r},\mathbf{r}')=(2\mu/\hbar^2)\sum_{JMls}\mathcal{Y}_{Jls}{}^M(\mathbf{r})\mathcal{Y}_{Jls}{}^{M*}(\mathbf{r}')\times r^{-1}r'^{-1}G_l(k;r,r'),\quad(2.4)$$

$$G_l(k;r,r')=(-)^{l+1}k^{-1}u_l(kr_<)w_l(kr_>),$$

$$\psi_0(\mathbf{k}s\nu,\mathbf{r})=(2\mu k/\pi\hbar^2)^{\frac{1}{2}}(kr)^{-1}\sum_{JMlm}i^lu_l(kr)\times\mathcal{Y}_{Jls}{}^M(\mathbf{r})Y_l{}^{m*}(\mathbf{k})C_{ls}(J,M;m,\nu)\quad(2.5)$$

$$\psi(\mathbf{k}s\nu,\mathbf{r})=(2\mu k/\pi\hbar^2)^{\frac{1}{2}}(kr)^{-1}\sum_{JMll'ms'}i^l\psi_{l's',ls}{}^J(k,r)\times\mathcal{Y}_{Jl's'}{}^M(\mathbf{r})Y_l{}^{m*}(\mathbf{k})C_{ls}(J,M;m,\nu),\quad(2.6)$$

where $C_{ls}(J,M;m,\nu)$ are the Clebsch-Gordan coefficients in the notation of Blatt and Weisskopf,[1] and

$$\mathcal{Y}_{Jls}{}^M(\mathbf{r})=\sum_{m\nu}C_{ls}(J,M;m,\nu)Y_l{}^m(\mathbf{r})\chi_s{}^\nu.$$

Furthermore, we have used the Riccatti-Bessel functions

$$\begin{aligned}u_l(z)&\equiv zj_l(z)=(\tfrac{1}{2}\pi z)^{\frac{1}{2}}J_{l+\frac{1}{2}}(z)=(-)^{l+1}u_l(-z)\\ v_l(z)&\equiv zn_l(z)=(\tfrac{1}{2}\pi z)^{\frac{1}{2}}N_{l+\frac{1}{2}}(z)=(-)^lv_l(-z)\\ w_l(z)&\equiv-v_l(z)-iu_l(z)=-izh_l{}^{(2)}(z)\\ &=-i(\tfrac{1}{2}\pi z)^{\frac{1}{2}}H_{l+\frac{1}{2}}{}^{(2)}(z)=(-)^lw_l(-z)^*,\end{aligned}\quad(2.7)$$

which are most convenient for solving the radial equation. Insertion in (2.2) leads to a set of coupled integral equations for the radial functions:

$$\psi_{l's',ls}{}^J(k,r)=u_l(kr)\delta_{ll'}\delta_{ss'}+\sum_{l''s''}\int_0^\infty dr'G_{l'}(k;r,r')\times V_{l's',l''s''}{}^J(r')\psi_{l''s'',ls}{}^J(k,r'),\quad(2.8)$$

where

$$V_{ls,l's'}{}^J(r)=\frac{2\mu}{\hbar^2}\int d\Omega\mathcal{Y}_{Jls}{}^{M*}(\mathbf{r})H_I(\mathbf{r})\mathcal{Y}_{Jl's'}{}^M(\mathbf{r}).\quad(2.9)$$

The meaning of the subscripts on ψ follows from (2.6) and (2.8). The first set "$l's'$" indicates the component of ψ belonging to specific orbital and spin angular momenta, while the second set, "ls" refers to the angular momenta of the incident beam, i.e., to the boundary condition.

The solution of (2.8) satisfies the set of differential equations

$$-\frac{d^2}{dr^2}\psi_{l's',ls}{}^J+\sum_{l''s''}V_{l's',l''s''}{}^J\psi_{l''s'',ls}{}^J+\frac{l(l+1)}{r^2}\psi_{l's',ls}{}^J=k^2\psi_{l's',ls}{}^J.\quad(2.10)$$

If we are considering the scattering of particles with no spin then V and ψ are diagonal and equations (2.8) and (2.10) become uncoupled. If H_I is invariant under space reflection then the conservation of parity implies that for the scattering of a spin $\frac{1}{2}$ particle by a spin-zero particle, the equations are also uncoupled. In case both particles have spin $\frac{1}{2}$ it is the tensor force alone which couples them.

The amplitude $\Theta_{s'\nu',s\nu}(\mathbf{k}',\mathbf{k})$ for scattering from the initial momentum $\hbar\mathbf{k}$ and spin $\hbar s,\hbar\nu$ to the final momentum $\hbar\mathbf{k}'$ and spin $\hbar s',\hbar\nu'$ is defined by the asymptotic form of ψ for large r; thus

$$\psi_+(\mathbf{k}s\nu,\mathbf{r})\sim[(\mu k)^{\frac{1}{2}}/\hbar(2\pi)^{\frac{3}{2}}]\times[\chi_s{}^\nu e^{i\mathbf{k}\cdot\mathbf{r}}+r^{-1}e^{ikr}\sum_{s'\nu'}\chi_{s'}{}^{\nu'}\Theta_{s'\nu',s\nu}(\mathbf{k}',\mathbf{k})],\quad(2.11)$$

where $\mathbf{k}'=k\mathbf{r}r^{-1}$. Taking the limit of (2.2) for large r

[1] J. M. Blatt and V. Weisskopf, *Theoretical Nuclear Physics* (John Wiley & Sons, Inc., New York, 1952).

leads by (2.3) to

$$\Theta_{s'\nu',s\nu}(\mathbf{k}',\mathbf{k})=-\frac{(2\pi)^2}{k}\int(d\mathbf{r})\psi_0^*(\mathbf{k}'s'\nu',\mathbf{r})\times H_I(\mathbf{r})\psi_+(\mathbf{k}s\nu,\mathbf{r}). \quad (2.12)$$

If we expand in spherical harmonics according to (2.5) and (2.6), we obtain

$$\Theta_{s'\nu',s\nu}(\mathbf{k}',\mathbf{k})=-\frac{4\pi}{k^2}\sum_{JMll'mm'}i^{l-l'}Y_{l'}^{m'}(\mathbf{k}')C_{l's'}(J,M;m',\nu')\times Y_l^{m*}(\mathbf{k})C_{ls}(J,M;m,\nu)\sum_{l''s''}\int_0^\infty dr u_{l'}(kr)\times V_{l's',l''s''}{}^J(r)\psi_{l''s'',ls}{}^J(k,r). \quad (2.13)$$

The scattering matrix is defined as the probability amplitude for finding, at the time $t=+\infty$, momentum $\hbar\mathbf{k}'$ and spins $\hbar s'$, $\hbar\nu'$, if they were $\hbar\mathbf{k}$ and $\hbar s$, $\hbar\nu$ at the time $t=-\infty$.[2,3]

$$\begin{aligned}(\mathbf{k}'s'\nu'|S|\mathbf{k}s\nu)&\equiv\lim_{t\to\infty}\int(d\mathbf{r})\psi_0^*(\mathbf{k}'s'\nu',\mathbf{r})e^{i(E'-E)t/\hbar}\psi_+(\mathbf{k}s\nu,\mathbf{r})\\&=\delta(E-E')[\delta(\Omega_k-\Omega_k')\delta_{ss'}\delta_{\nu\nu'}+(ik/2\pi)\Theta_{s'\nu',s\nu}(\mathbf{k}',\mathbf{k})]\\&\equiv\delta(E-E')\sum_{JMll'mm'}Y_{l'}^{m'}(\mathbf{k}')Y_l^{m*}(\mathbf{k})i^{l-l'}\\&\quad\times C_{l's'}(J,M;m',\nu')C_{ls}(J,M;m,\nu)S_{l's',ls}{}^J(k),\end{aligned} \quad (2.14)$$

where the second line follows from the integral equation (2.2) and (2.3). It follows from (2.12) that

$$\Theta_{s'\nu',s\nu}(\mathbf{k}',\mathbf{k})=-2\pi ik^{-1}\sum_{JMll'mm'}Y_{l'}^{m'}(\mathbf{k}')Y_l^{m*}(\mathbf{k})\times C_{l's'}(J,M;m',\nu')C_{ls}(J,M;m,\nu)i^{l-l'}\times(S_{l's',ls}{}^J-\delta_{ll'}\delta_{ss'}). \quad (2.15)$$

Conservation of particles implies that S is unitary. It therefore follows from (2.15) that

$$-2\pi ik^{-1}[\Theta_{s'\nu',s\nu}(\mathbf{k}',\mathbf{k})-\Theta_{s\nu,s'\nu'}{}^*(\mathbf{k},\mathbf{k}')]=\sum_{s''\nu''}\int d\Omega_k''\Theta_{s''\nu'',s\nu}(\mathbf{k}'',\mathbf{k})\Theta_{s''\nu'',s'\nu'}{}^*(\mathbf{k}'',\mathbf{k}'). \quad (2.16)$$

A special case is the "optical theorem," which is obtained by setting $s=s'$, $\nu=\nu'$, $\mathbf{k}=\mathbf{k}'$[4]:

$$4\pi k^{-1}\,\mathrm{Im}\Theta_{s\nu,s\nu}(\mathbf{k},\mathbf{k})=\sum_{s'\nu'}\int d\Omega_k'|\Theta_{s'\nu',s\nu}(\mathbf{k}',\mathbf{k})|^2=\sigma_{s\nu}{}^{\mathrm{total}}(\mathbf{k}). \quad (2.16')$$

[2] J. Jauch and F. Rohrlich, *Theory of Photons and Electrons* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1955).
[3] C. Møller, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **23**, No. 1 (1945).
[4] We write Re A and Im A for the real and imaginary parts of A.

A further property of the S matrix follows if H_I is invariant under time reversal. We use a time-reversal operator[5] ϑ and spin functions and spherical harmonics such that

$$\vartheta\mathcal{Y}_{Jls}{}^M(\mathbf{r})=(-)^{J+M}\mathcal{Y}_{Jls}{}^{-M}(\mathbf{r}).$$

This is obtained by taking real χ and spherical harmonics which are such that[6]

$$Y_l^{m*}=(-)^{l+m}Y_l^{-m},$$

and the time-reversal operator

$$\vartheta=(i\sigma_y{}^{(1)})(i\sigma_y{}^{(2)})K,$$

$\sigma^{(1)}$ and $\sigma^{(2)}$ being the spin matrices for particles #1 and #2 (with $i\sigma_y=1$ if the particle has spin zero) and K, the antiunitary complex conjugation operator. The Clebsch-Gordan coefficients are such that

$$C_{ls}(J,-M;-m,-\nu)=(-)^{l+s-J}C_{ls}(J,M;m,\nu).$$

With these conventions we have

$$\vartheta\psi_0(\mathbf{k}s\nu,\mathbf{r})=(-)^{s+\nu}\psi_0(-\mathbf{k}s-\nu;\mathbf{r}),$$

and therefore, by (2.2),

$$\vartheta\psi_+(\mathbf{k}s\nu,\mathbf{r})=(-)^{s+\nu}\psi_-(-\mathbf{k}s-\nu,\mathbf{r}),$$

where ψ_- satisfies the integral equation (2.2) with $G_-=G_+^*$, the incoming wave Green's function.

It then follows from

$$(\psi_1,\psi_2)=(\vartheta\psi_2,\vartheta\psi_1)$$

and the assumed time-reversal invariance of H_I that the potential matrix of (2.9) is *symmetric:*

$$V_{ls,l's'}{}^J(r)=V_{l's',ls}{}^J(r). \quad (2.17)$$

Since H_I is Hermitian, $V_{ls,l's'}{}^J$ is consequently *real.* For the scattering amplitude, we get, from (2.12),

$$\Theta_{s'\nu',s\nu}(\mathbf{k}',\mathbf{k})=(-)^{s-s'+\nu-\nu'}\Theta_{s-\nu,s'-\nu'}(-\mathbf{k},-\mathbf{k}'), \quad (2.18)$$

which is the *reciprocity theorem.* It follows from (2.15) that it is equivalent to the *symmetry* of $S_{ls,l's'}{}^J$ as defined in (2.14):

$$S_{ls,l's'}{}^J=S_{l's',ls}{}^J. \quad (2.19)$$

S^J being unitary and symmetric, it can be diagonalized by an orthogonal real matrix U:

$$S_{ls,l's'}{}^J=\sum_\alpha U_{ls,\alpha}{}^J\exp(2i\delta_\alpha{}^J)U_{\alpha,l's'}{}^J, \quad (2.20)$$

where the $\delta_\alpha{}^J$ are *real.*

Comparison of (2.13) with (2.15) gives us another expression for the S matrix:

$$S_{l's',ls}{}^J(k)=\delta_{ll'}\delta_{ss'}-2ik^{-1}\sum_{l''s''}\int_0^\infty dr u_{l'}(kr)\times V_{l's',l''s''}{}^J(r)\psi_{l''s'',ls}{}^J(k,r). \quad (2.21)$$

[5] E. P. Wigner, *Group Theory* (Academic Press, Inc., New York, 1959), p. 325 ff.
[6] Those of Blatt and Weisskopf,[1] say, multiplied by i^l; see footnote 5, p. 345, of E. P. Wigner.[5]

We may now identify S^J by the asymptotic form of the radial wave function. In order to do that we require the asymptotic values of the Riccatti-Bessel functions for large r:

$$\begin{aligned} u_l(kr) &\sim \sin(kr-\tfrac{1}{2}\pi l),\\ v_l(kr) &\sim -\cos(kr-\tfrac{1}{2}\pi l),\\ w_l(kr) &\sim i^l e^{-ikr}. \end{aligned} \tag{2.22}$$

Equation (2.8) together with (2.4) and (2.21) shows that as $r \to \infty$,

$$\psi_{ls,l's'}{}^J(k,r) \sim \tfrac{1}{2} i^{l+1}[\delta_{ll'}\delta_{ss'}e^{-ikr} - (-)^l e^{ikr} S_{ls,l's'}{}^J(k)], \tag{2.23}$$

and therefore, by (2.20),

$$\psi_{ls,\alpha}{}^J(k,r) \equiv \sum_{l's'} \psi_{ls,l's'}{}^J(k,r) U_{l's',\alpha}{}^J(k) \sim U_{ls,\alpha}{}^J(k) \exp[i\delta_\alpha{}^J(k)] \sin(kr-\tfrac{1}{2}\pi l+\delta_\alpha{}^J). \tag{2.24}$$

Thus $\delta_\alpha{}^J(k)$ is identified as the eigenphaseshift. The characteristic property of $\psi_{ls,\alpha}{}^J$ is that all its components experience the same phaseshift.

We want to investigate the properties of solutions of the radial equation (2.10). In Secs. 3 to 8 we restrict ourselves to the case of no coupling, which is realized when one of the particles has spin zero and the other, spin less than two (provided H_I conserves parity); or else if both have spin $\frac{1}{2}$, but tensor forces are neglected.

If there is no coupling; i.e., the matrix V^J is diagonal, then only one subscript, l, will be used everywhere, that being the only index on which Eq. (2.10) explicitly depends via the centrifugal term.

3. REGULAR AND IRREGULAR SOLUTIONS

We return to the radial equation (2.10) in the case of no coupling. Rather than considering the "physical" solution ψ_l we define regular and irregular solutions by boundary conditions which lead to simple properties as functions of k.

The weakest assumptions we shall ever make concerning the potential are the existence of its first and second absolute moments:

$$\int_0^\infty dr r|V(r)| < \infty, \tag{3.1a}$$

$$\int_0^\infty dr r^2|V(r)| < \infty. \tag{3.1b}$$

Whenever stronger assumptions are made they will be stated explicitly.

Hypothesis (3.1a) implies that V behaves better than r^{-2} near the origin. There exists consequently a regular solution $\varphi_l(k,r)$ of (2.10) which near $r=0$ behaves like $u_l(kr)$. As $kr \to 0$ we have[7]

$$\begin{aligned} u_l(kr) &= (kr)^{l+1}/(2l+1)!!+O[(kr)^{l+3}],\\ v_l(kr) &= -(kr)^{-l}(2l-1)!!+O[(kr)^{-l+2}]. \end{aligned} \tag{3.2}$$

We therefore define $\varphi_l(k,r)$ by the boundary condition

$$\lim_{r\to 0}(2l+1)!!r^{-l-1}\varphi_l(k,r)=1. \tag{3.3}$$

It then follows immediately that $\varphi_l(k,r)$ is a function of k^2 only and that for real k it is real.

Hypothesis (3.1b) implies that at infinity V behaves better than r^{-3} so that a Coulomb field, for example, is excluded. It follows (as will be shown later) that at infinity all solutions of (2.10) oscillate like sine or cosine waves. It is then convenient to define another solution $f_l(k,r)$ by the boundary condition

$$\lim_{r\to\infty} e^{ikr} f_l(k,r) = i^l. \tag{3.4}$$

This function does not vanish at $r=0$, in general, but it is $O(r^{-l})$ there, as is $w_l(kr)$. It follows immediately from the boundary condition (3.4) and from the reality of the differential equation (2.10) that for real k

$$f_l^*(-k,r) = (-)^l f_l(k,r). \tag{3.5}$$

We now want to extend all our definitions to complex values of k. It then follows from the k independence of the boundary condition[8] (3.3) that for fixed r, $\varphi_l(k,r)$ is an analytic function of k regular for all finite values of k; i.e., an entire function of k. The function $f_l(k,r)$ is for fixed $r>0$ an analytic function of k regular in the open lower half of the complex k plane; in the upper half of the k plane it may be expected to have singularities since (3.4) is not sufficient there to define $f_l(k,r)$ uniquely. These statements are intended merely as a guide and will be proved later.

It is clear that in any region of analyticity connected with the real axis Eq. (3.5) implies

$$f_l^*(-k^*,r) = (-)^l f_l(k,r). \tag{3.5'}$$

We can readily replace the differential equation (2.10) and boundary conditions (3.3) or (3.4) by integral equations. If we define

$$\begin{aligned} g_l(k;r,r') &\equiv k^{-1}[u_l(kr')v_l(kr)-u_l(kr)v_l(kr')]\\ &= i(-)^l(2k)^{-1}[w_l(kr)w_l(-kr')-w_l(-kr)w_l(kr')], \end{aligned} \tag{3.6}$$

[7] We use the following notation: "$f(x)=O(x)$ as $x\to\infty$ (or 0)" means that $f(x)/x$ is bounded as $x\to\infty$ (or 0); "$f(x)=o(x)$ as $x\to\infty$ (or 0)" means that $f(x)/x$ tends to naught as $x\to\infty$ (or 0).

[8] According to a theorem by Poincaré the solution of an ordinary linear differential equation containing an entire function of a parameter k, defined by a boundary condition independent of k, is itself an entire function of k. See footnote reference 9; also see footnote 8 of Jost and Pais.[10] We shall use this theorem as a guidance only and prove it for the special case of $\varphi_l(k,r)$.

[9] E. Hilb, *Encycl. der Math. Wissensch.* (B. G. Teubner, Leipzig, 1915), Vol. 2, Part 2, p. 501.

[10] R. Jost and A. Pais, Phys. Rev. 82, 840 (1951).

then

$$G_l^{(1)}(k;r,r')=\begin{cases} g_l(k;r,r'), & r'\leq r, \\ 0, & r'\geq r, \end{cases}$$

is a Green's function [compare with (2.4)] and so is

$$G_l^{(2)}(k;r,r')=\begin{cases} 0, & r'\leq r, \\ -g_l(k;r,r'), & r'\geq r. \end{cases}$$

The first is appropriate to the definition of φ_l and the second, to that of f_l; thus

$$\varphi_l(k,r)=k^{-l-1}u_l(kr) + \int_0^r dr' g_l(k;r,r')V(r')\varphi_l(k,r'), \quad (3.7)$$

$$f_l(k,r)=w_l(kr)-\int_r^\infty dr' g_l(k;r,r')V(r')f_l(k,r'). \quad (3.8)$$

The existence and analytic properties of φ_l and f_l are proved by means of these integral equations. Their advantage over the integral equation (2.8) for the physical wave function ψ_l is that they can be solved by successive approximations, provided only that V satisfies (3.1), irrespective of its *strength*. The reason is that the integrations run from naught to r only, or from r to infinity.

In order to prove the convergence[11] of the sequence of successive approximations, one uses the following bounds, true in the entire complex plane[16]

$$\begin{aligned} |u_l(kr)| &\leq Ce^{|\nu|r}[L(|k|r)]^{l+1}, \\ |v_l(kr)| &\leq Ce^{|\nu|r}[L(|k|r)]^{-l}, \\ |w_l(kr)| &\leq Ce^{\nu r}[L(|k|r)]^{-l}, \end{aligned} \quad (3.9)$$

where $\nu\equiv \mathrm{Im}k$ and

$$L(x)\equiv x/(1+x).$$

It is then easily seen that for $r'\leq r$

$$|g_l(k;r,r')|=|g_l(k;r',r)| \leq Ce^{|\nu|(r-r')}|k|^{-1}[L(|k|r)]^{l+1}[L(|k|r')]^{-l}. \quad (3.10)$$

We now solve (3.7) by successive approximation:

$$\varphi_l(k,r)=\sum_0^\infty \varphi_l^{(n)}(k,r),$$

where

$$\varphi_l^{(0)}(k,r)=k^{-l-1}u_l(kr)$$

$$\varphi_l^{(n)}(k,r)=\int_0^r dr' g_l(k;r,r')V(r')\varphi_l^{(n-1)}(k,r'), \quad n\geq 1.$$

If we use (3.10), we get

$$|\varphi_l^{(n)}(k,r)|\leq C\int_0^r dr' e^{|\nu|(r-r')}|k|^{-1}[L(|k|r)]^{l+1} \times[L(|k|r')]^{-l}|V(r')||\varphi_l^{(n-1)}(k,r')|.$$

Now writing for the moment,

$$\mathfrak{A}_l^{(n)}(k,r)\equiv \varphi_l^{(n)}(k,r)e^{-|\nu|r}|k|^{l+1}[L(|k|r)]^{-l-1},$$

we have

$$|\mathfrak{A}_l^{(0)}(k,r)|\leq C$$

$$|\mathfrak{A}_l^{(n)}(k,r)|\leq C\int_0^r dr' |\mathfrak{A}_l^{(n-1)}(k,r')| \times|V(r')|L(|k|r')|k|^{-1}, \quad n\geq 1,$$

and therefore

$$|\mathfrak{A}_l^{(n)}(k,r)|\leq C^{n+1}\int_0^r dr_1\cdots\int_0^{r_{n-1}} dr_n |V(r_1)| \times\frac{r_1}{1+|k|r_1}\cdots|V(r_n)|\frac{r_n}{1+|k|r_n} =\frac{C^{n+1}}{n!}\left[\int_0^r dr'|V(r')|\frac{r'}{1+|k|r'}\right]^n,$$

so that

$$\left|\sum_0^\infty \mathfrak{A}_l^{(n)}(k,r)\right|\leq C\exp\left[C\int_0^r dr'|V(r')|r'(1+|k|r')^{-1}\right].$$

As a result the series $\sum\varphi_l^{(n)}$ converges absolutely and uniformly for all r and in every finite region in the complex k plane. Furthermore, we find that

$$|\varphi_l(k,r)|\leq Ce^{|\nu|r}\left(\frac{r}{1+|k|r}\right)^{l+1} \times\exp\left[C\int_0^r dr'|V(r')|\frac{r'}{1+|k|r'}\right]. \quad (3.11)$$

Since g_l and $\varphi_l^{(0)}$ are entire analytic functions of k, so is each $\varphi_l^{(n)}$. It then follows that for every fixed r, $\varphi_l(k,r)$ is an entire function of k^2.

We may now insert (3.11) in the integral equation (3.7) and obtain the inequality

$$|\varphi_l(k,r)-k^{-l-1}u_l(kr)| \leq Ce^{|\nu|r}\left(\frac{r}{1+|k|r}\right)^{l+1}\int_0^r dr'|V(r')|\frac{r'}{1+|k|r'}. \quad (3.12)$$

[11] The procedure below follows Jost[12] and Levinson.[13] It can be generalized to certain restricted nonlocal potentials; see Martin.[14,15]
[12] R. Jost, Helv. Phys. Acta **20**, 256 (1947).
[13] N. Levinson, Kgl. Danske Videnskab. Selskab., Mat.-fys. Medd. **25**, No. 9 (1949).
[14] A. Martin, Compt. rend. **243**, 22 (1956).
[15] A. Martin, Nuovo cimento **14**, 403 (1959).
[16] The first inequality was given by Levinson,[13] the others by Newton.[17]
[17] R. G. Newton, Phys. Rev. **100**, 412 (1955).

The integral on the right-hand side tends to naught as $|k| \to \infty$, even if at $r=0$ only the first moment of $|V|$ exists. That is seen by writing

$$\int_0^\infty dr|V(r)|\frac{r}{1+|k|r}=\int_0^a+\int_a^\infty \leq \int_0^a drr|V(r)| + \int_a^\infty dr|V(r)|\,|k|^{-1}.$$

Hence if we choose a and k so that

$$\int_0^a drr|V(r)| \leq \tfrac{1}{2}\epsilon, \quad |k| \geq \int_a^\infty dr|V(r)|\cdot 2\epsilon^{-1}$$

then

$$\int_0^\infty dr|V(r)|r(1+|k|r)^{-1} \leq \epsilon.$$

The right-hand side of (3.12) is therefore

$$o(|k|^{-l-1}e^{|\nu|r}) \quad \text{as} \quad |k| \to \infty,$$

and consequently, as $|k| \to \infty$

$$\varphi_l(k,r)=k^{-l-1}\sin(kr-\tfrac{1}{2}\pi l)+o(|k|^{-l-1}e^{|\nu|r}) \qquad (3.13)$$

uniformly in r. It is clear from (3.12) that if V is absolutely integrable, then the remainder is $O(|k|^{-l-2}e^{|\nu|r})$.

A similar procedure is followed for the function $f_l(k,r)=\sum f_l^{(n)}(k,r)$, where

$$f_l^{(0)}(k,r)=w_l(kr)$$

$$f_l^{(n)}(k,r)=-\int_r^\infty dr' g_l(k;r,r')V(r')f_l^{(n-1)}(k,r'), \quad n\geq 1.$$

One then finds that the series $\sum h_l^{(n)}(k,r)$, where

$$h_l^{(n)}(k,r) \equiv e^{-\nu r}[L(|k|r)]^l f_l^{(n)}(k,r),$$

is dominated by a series which can be summed to

$$C\exp\left[C\int_r^\infty dr'|V(r')| \times\exp[(\nu+|\nu'|)(r'-r)]r'(1+|k|r')^{-1}\right].$$

The series $\sum f_l^{(n)}$ therefore converges uniformly for all $r\geq r_0>0$ and for any closed region in the complex k plane not including $k=0$, where

$$\alpha \equiv \int_0^\infty dr|V(r)|re^{(\nu+|\nu|)r}$$

is finite. Thus $f_l(k,r)$ exists, is continuous, and is obtainable by successive approximations from (3.8) for all $r>0$ and all finite $k\neq 0$ in the lower half-plane, including the real axis, provided only that V possesses a finite first absolute moment. If, moreover, V decreases exponentially at infinity so that

$$\int_0^\infty drr|V(r)|e^{2ar}<\infty \qquad (3.14)$$

for some $a>0$, then it follows that $f_l(k,r)$ exists and is continuous (and is obtainable by successive approximations) in a strip in the upper half of the complex k plane with $\text{Im}k\leq a$, except at $k=0$.

We also get the inequality

$$|f_l(k,r)| \leq Ce^{\nu r}[(1+|k|r)/|k|r]^l e^{C\alpha}, \qquad (3.15)$$

which inserted in (3.8) yields

$$|f_l(k,r)-w_l(kr)| \leq Ce^{\nu r}\left(\frac{1+|k|r}{|k|r}\right)^l \times e^{C\alpha}\int_r^\infty dr'|V(r')|e^{(\nu+|\nu|)(r'-r)}\frac{r'}{1+|k|r'}. \qquad (3.16)$$

By the same argument that follows (3.12) the right-hand side of (3.16) is $o(e^{\nu r})$ as $|k| \to \infty$ uniformly for all $r\geq r_0>0$ in the lower half of the complex k plane including the real axis, and in a strip of width a in the upper half-plane if (3.14) holds. Therefore, as $|k| \to \infty$

$$f_l(k,r)=i^l e^{-ikr}+o(e^{\nu r}). \qquad (3.17)$$

The inequality (3.16) also shows that

$$\lim_{k\to 0} k^l f_l(k,r)$$

exists for all finite $r>0$ if the limit is carried out in the region of regularity.

In order to show that $f_l(k,r)$ is an analytic function of k we must show the existence and continuity of its first derivative with respect to k in the same manner as those of $f_l(k,r)$ itself. (Since the integral in (3.8) converges absolutely, differentiation under the integral sign is permitted.) We cannot use the same argument here as for $\varphi_l(k,r)$ because $f_l^{(n)}$ is not necessarily regular. The result is that, provided V has a finite second absolute moment, $f_l(k,r)$ for fixed $r>0$ is an analytic function of k regular everywhere in the open lower half of the complex plane and continuous on the real axis, except at $k=0$. If the potential satisfies (3.14) then the region of regularity includes a strip in the upper half-plane up to $\text{Im}k<a$, except for a pole of order l at $k=0$. If the potential vanishes identically outside a finite region, then $k^l f_l(k,r)$ is an entire function of k for all fixed $r>0$.

If the potential satisfies (3.14) then one may obtain information about the singularities of $f_l(k,r)$ for $\text{Im}k\geq a$ in a relatively simple way.[18] If we write

$$f_l(k,r) \equiv f_l^{(0)}(k,r)+\chi_l^{(1)}(k,r),$$

where $f_l^{(0)}(k,r)=w_l(kr)$, then $\chi_l^{(1)}$ satisfies the integral

[18] T. Regge, Nuovo Cimento 9, 295 (1958).

equation

$$\chi_l^{(1)}(k,r)=f_l^{(1)}(k,r)-\int_r^\infty dr' g_l(k;r,r')V(r')\chi_l^{(1)}(k,r'),$$

where $f_l^{(1)}$ is simply the first Born approximation:

$$f_l^{(1)}(k,r)=-\int_r^\infty dr' g_l(k;r,r')V(r')w_l(kr').$$

Suppose that for a given V this integral is carried out and it admits an analytic continuation into a region in the upper half-plane with $\text{Im}k\geq a$, and we can set there

$$|f_l^{(1)}(k,r)|\leq Ce^{\gamma r}.$$

We may then use this inequality in place of (3.9) and prove the analyticity of $\chi_l^{(1)}$ by the same arguments which prove it for f_l. The result is evidently that $\chi_l^{(1)}(k,r)$ is regular where $f_l^{(1)}(k,r)$ is in the region $\text{Im}k<2a-\gamma$. If, for example, $\gamma=\nu-2a$ as it is for $\text{Im}k<a$, then the continuation works for $\text{Im}k=\nu<2a$. One may repeat the same argument by examining explicitly the analytic continuation of the second Born approximation $f_l^{(2)}$, and thus extend the strip of analyticity further and further, except for explicitly isolated singularities.

4. JOST FUNCTION $f_l(k)$

The functions $f_l(k,r)$ and $f_l(-k,r)$ are two linearly independent solutions (for $k\neq 0$) of the differential equation (2.10). The regular solution $\varphi_l(k,r)$ can therefore be expressed as a linear combination of them. Since $\varphi_l(k,r)$ is even in k this defines a function $f_l(k)$, so that[12]

$$\varphi_l(k,r)=\tfrac{1}{2}ik^{-l-1}\times[f_l(-k)f_l(k,r)-(-)^l f_l(k)f_l(-k,r)]. \quad (4.1)$$

We want to get a more explicit equation for $f_l(k)$. That can be obtained by taking the Wronskian of $\varphi_l(k,r)$ and $f_l(k,r)$. The Wronskian of two solutions of the same linear second-order differential equation being independent of r, we readily find, by evaluating it at $r\to\infty$ and using the boundary condition (3.4) that

$$W[f_l(k,r),f_l(-k,r)]=(-)^l 2ik,$$

where

$$W[f,g]\equiv fg'-f'g. \quad (4.2)$$

If we make use of this and (4.1), we obtain

$$f_l(k)=k^l W[f_l(k,r),\varphi_l(k,r)]. \quad (4.3)$$

Because of the boundary condition (3.3) this implies that

$$f_l(k)=\lim_{r\to 0}(kr)^l f_l(k,r)/(2l-1)!!. \quad (4.3')$$

If we insert the integral equations (3.7) and (3.8) in (4.3) and evaluate it at $r=0$ or $r=\infty$, we obtain the following two integral representations for $f_l(k)$:

$$f_l(k)=1+k^{-l}\int_0^\infty dr f_l(k,r)V(r)u_l(kr)$$
$$=1+k^l\int_0^\infty dr\varphi_l(kr)V(r)w_l(kr). \quad (4.4)$$

We may now express the S matrix in terms of $f_l(k)$. The asymptotic form for large r of $\varphi_l(k,r)$ follows immediately from (4.1) and the boundary condition (3.4); thus

$$\varphi_l(k,r)\sim\tfrac{1}{2}i^{l+1}k^{-l-1}\times[f_l(-k)e^{-ikr}-(-)^l f_l(k)e^{ikr}]. \quad (4.1')$$

Comparison with (2.23) shows that

$$S_l(k)=f_l(k)/f_l(-k). \quad (4.5)$$

It follows that

$$S_l(-k)=1/S_l(k). \quad (4.6)$$

Furthermore, φ_l being real and even in k it follows from (3.5) and (4.3) that for real k

$$f_l^*(-k)=f_l(k), \quad (4.7)$$

and, consequently,

$$|S_l(k)|=1.$$

This is the unitarity condition.

The relation between φ_l and the physical wave function ψ_l is provided by a comparison of (4.1′) with (2.23); thus

$$\psi_l(k,r)=[k^{l+1}/f_l(-k)]\varphi_l(k,r). \quad (4.8)$$

This furnishes the physical significance of the function $f_l(k)$. Equation (4.5) together with (4.7) and (2.20) shows that

$$f_l(k)=|f_l(k)|\exp[i\delta_l(k)], \quad (4.9)$$

where δ_l is the *phaseshift* for the lth partial wave, while (4.8) with (3.3) shows that

$$|\psi_l(k,r)|^2\underset{r\to 0}{\longrightarrow}|\psi_l^{(0)}(k,r)|^2/|f_l(k)|^2,$$

$\psi_l^{(0)}$ being the wave function in the absence of a potential. Thus the phase of $f_l(k)$ is the lth phaseshift and the inverse of the square of its modulus measures the probability of finding the particles in each other's proximity relative to what it would be in the absence of forces between them.

The Jost function $f_l(k)$ may also be approached from quite a different point of view. Suppose that one were to solve the integral equation (2.8) for the physical wave function by the Fredholm method.[10,19] One would then have to form the Fredholm determinant

$$\Delta_l(k)=1+\sum_{n=1}^\infty\frac{(-)^n}{n!}\int_0^\infty dr_1\cdots\int_0^\infty dr_n \times V(r_1)\cdots V(r_n)D_l(k;r_1,\cdots,r_n),$$

where

$$D_l(k;r_1,\cdots,r_n)=\begin{vmatrix} G_l(k;r_1,r_1) & G_l(k;r_1,r_2)\cdots \\ G_l(k;r_2,r_1) & G_l(k;r_2,r_2)\cdots \\ \vdots & \end{vmatrix}.$$

$G_l(k;r,r')$ being given by (2.4). Because the integrand in $\Delta_l(k)$ is symmetric in $r_1,\cdots,r_n$, and because

[19] E. T. Whittaker and G. N. Watson, *Modern Analysis* (Cambridge University Press, New York, 1948), p. 211 ff.

of the symmetry properties of the spherical Bessel functions we can write

$$\Delta_l(-k)=1+\sum_1^\infty k^{-n}\int_0^\infty dr_1\int_0^{r_1}dr_2\cdots\int_0^{r_{n-1}}dr_n \times V(r_1)\cdots V(r_n)d_l(k;r_1,\cdots,r_n) \quad (4.10)$$

with

$$d_l(k;r_1,\cdots,r_n) = \begin{vmatrix} w_l(kr_1)u_l(kr_1) & w_l(kr_1)u_l(kr_2)\cdots w_l(kr_1)u_l(kr_n) \\ w_l(kr_1)u_l(kr_2) & w_l(kr_2)u_l(kr_2)\cdots w_l(kr_2)u_l(kr_n) \\ \vdots & \\ w_l(kr_1)u_l(kr_n) & w_l(kr_2)u_l(kr_n)\cdots w_l(kr_n)n_l(kr_n)\end{vmatrix} \equiv w_l(kr_1)s_l(k;r_1;r_2,\cdots,r_n).$$

Consequently,

$$\Delta_l(-k)=1+k^{-1}\int_0^\infty dr w_l(kr)V(r)g_l(kr), \quad (4.11)$$

where

$$g_l(k,r)=u_l(kr)+\sum_1^\infty k^{-n}\int_0^r dr_1\int_0^{r_1}dr_2\cdots\int_0^{r_{n-1}}dr_n \times V(r_1)\cdots V(r_n)s_l(k;r;r_1,\cdots,r_n).$$

Now since $s_l\,(k;r;r,r_2,\cdots,r_n)=0$ and

$$W[w_l(kr),u_l(kr)]=k,$$

we readily find that

$$W[w_l(kr),g_l(k,r)]=k\left[1+\sum_1^\infty k^{-n}\int_0^r dr_1\cdots\int_0^{r_{n-1}}dr_n \times V(r_1)\cdots V(r_n)d_l(k;r_1,\cdots,r_n)\right]. \quad (4.12)$$

Differentiation with respect to r yields

$$w_l(kr)g_l''(k,r)-g_l(k,r)w_l''(kr)=kw_l(kr)V(r)g_l(k,r),$$

which, because of the differential equation satisfied by $w_l(kr)$, shows that $g_l(k,r)$ solves the differential equation (2.10). Furthermore one easily sees from (4.12) that as $r\to 0$,

$$g_l(k,r)=u_l(kr)+o(r^{l+1}).$$

It follows that

$$g_l(k,r)=k^{l+1}\varphi_l(k,r), \quad (4.13)$$

and therefore by (4.11) and (4.3),

$$f_l(k)=\Delta_l(-k). \quad (4.14)$$

Thus the function $f_l(-k)$ is the Fredholm determinant of Eq. (2.8).[10]

We now want to examine the analytic properties of $f_l(k)$ in the complex plane; that will, via (4.5), give us information on the analytic properties of the S matrix.

In order to extend $f_l(k)$ into the complex plane we may use either (4.3) or (4.4). The inequalities (3.9) and (3.11) lead from the second version of (4.4) to

$$|f_l(k)-1|\leq C\int_0^\infty dr|V(r)|e^{(\nu+|\nu|)r}r(1+|k|r)^{-1}. \quad (4.15)$$

Under the assumptions (3.1) on the potential, the integral thus converges absolutely so that $f_l(k)$ exists and is continuous for all k in the closed lower half-plane. If the potential also satisfies (3.14) then the same conclusion can be drawn in an additional strip in the upper half-plane with $\mathrm{Im}k\leq a$. Because of the absolute convergence we may differentiate (4.4) with respect to k under the integral sign and then use inequalities obtained from differentiating (3.8) with respect to k. The result is that under the hypothesis (3.1) $f_l(k)$ is an analytic function of k regular in the open lower half of the complex plane and continuous on the real axis. If the potential fulfills (3.14) then $f_l(k)$ is analytic also in a strip in the upper half-plane with $\mathrm{Im}k<a$. If the potential vanishes asymptotically faster than *every* exponential (e.g., if it has a gaussian tail or if it vanishes identically beyond a finite radius) then $f_l(k)$ is an entire function of k. In any region of analyticity connected with the real axis (4.7) leads to

$$f_l^*(-k^*)=f_l(k). \quad (4.7')$$

The arguments at the end of Sec. 3, which in specific cases of (3.14) may allow the analytic continuation of $f_l(k,r)$ beyond $\mathrm{Im}k=a$ by examination of the terms in the successive approximations,[18] are now applicable to $f_l(k)$. It may thus be possible in many practical cases to use the first Born approximation in order to extend the analytic continuation to $\mathrm{Im}k<2a$, the second, to $\mathrm{Im}k<3a$, etc.

We now want to examine the behavior of $f_l(k)$ as $k\to\infty$. The inequality (4.15) tells us directly that for $\mathrm{Im}k\leq 0$,

$$\lim_{|k|\to\infty} f_l(k)=1. \quad (4.16)$$

In fact we can conclude from (4.3), (3.17), and (2.22) that when $\mathrm{Im}k\leq 0$, as $|k|\to\infty$,

$$f_l(k)=1+(2ik)^{-1}\int_0^\infty drV(r)+o(k^{-1}), \quad (4.16')$$

provided that $V(r)$ is integrable at $r=0$; otherwise the second term need be only $o(1)$. Equation (4.16′) says that at very high energies the Born approximation for $f_l(k)$ is good.[20]

In the upper half-plane we can draw interesting conclusions only if for $r>R$ the potential vanishes identi-

[20] The analyticity properties of $f_l(k)$ together with (4.7) and (4.16) imply, of course, that $f_l(k)$ satisfies a simple "dispersion relation" obtainable immediately from Cauchy's theorem. This was pointed out explicitly by Giambiagi and Kibble,[21] but it does not appear to have any useful application.

[21] J. J. Giambiagi and T. W. B. Kibble, Ann. Phys. 7, 39 (1959).

cally. In that event (4.15) shows that

$$|(f_l(k)-1)e^{2ikR}|\leq C\int_0^R dr|V(r)|r(1+|k|r)^{-1}.$$

This implies that for $\mathrm{Im}k>0$

$$(f_l(k)-1)e^{2ikR}=o(1) \quad \text{as} \quad |k|\to\infty \tag{4.17}$$

if the potential has a finite first absolute moment; if it is also absolutely integrable then the right-hand side of (4.17) is $O(|k|^{-1})$.

Next we want to look at the zeros of $f_l(k)$. Suppose that $f_l(k)$ vanishes at a point in the lower half-plane:

$$f_l(k_0)=0, \quad \mathrm{Im}k_0<0.$$

Since by (4.3) this means that the Wronskian between $f_l(k_0,r)$ and $\varphi_l(k_0,r)$ is zero, the two solutions are multiples of one another:

$$f_l(k_0,r)=c\varphi_l(k_0,r). \tag{4.18}$$

But k_0 being in the lower half-plane, the left-hand side of (4.18) decreases exponentially at infinity, while the right-hand side vanishes at the origin. Consequently, both sides are square integrable and k_0^2 is a discrete eigenvalue of the Schrödinger equation; there is a bound state of energy $\hbar^2k_0^2/2\mu$.

It is proved by the standard method that the eigenvalues k_0^2 must be real. If k_0 is a root of $f_l(k)$ in the lower half-plane, then by (4.7′), so is $-k_0^*$. Multiplication of the Schrödinger equation for $\varphi_l(k,r)$ by $\varphi_l(k',r)$ and subtraction from that for $\varphi_l(k',r)$ multiplied by $\varphi_l(k,r)$ leads to

$$(d/dr)W[\varphi_l(k,r),\varphi_l(k',r)] =(k^2-k'^2)\varphi_l(k,r)\varphi_l(k',r). \tag{4.19}$$

If we now set $k=k_0$, $k'=-k_0^*$ and integrate from $r=0$ to $r=\infty$, we get

$$\mathrm{Im}k_0^2\int_0^\infty dr|\varphi_l(k_0,r)|^2=0$$

and therefore k_0^2 must be real.

The converse is also true. If k_0^2 is a discrete eigenvalue then $f_l(k_0,r)$ must vanish at $r=0$ when k_0 is taken in the lower half plane. Hence by (4.3′) $f_l(k_0)=0$.

If $k_0=-i\kappa$, $\kappa>0$, is a root of $f_l(k)$ and $k=i\kappa$ lies in a region of analyticity of $f_l(k)$ connected with the real axis, then Eq. (4.1) at once yields the following value for the constant c of (4.18):

$$c=-2i^{-l}\kappa^{l+1}/f_l(i|\kappa|). \tag{4.20}$$

In general, however, we cannot draw this conclusion.

The function $f_l(k)$ cannot have any roots on the real axis, except possibly at $k=0$. That follows at once from the fact that by (4.7) $f_l(-k)$ vanishes when $f_l(k)$ does for real k. Equation (4.1) then shows that $\varphi_l(k,r)$ would vanish identically in r. Since that contradicts the boundary condition (3.3), $f_l(k)$ cannot vanish for real $k\neq0$.

It is possible for $f_l(k)$ to be zero for $k=0$. We then conclude that the function

$$h_l(k,r)\equiv k^lf_l(k,r)$$

is, for $k=0$, a multiple of $\varphi_l(0,r)$

$$h_l(0,r)=c\varphi_l(0,r). \tag{4.18′}$$

For $l=0$, however, the boundary condition (3.4) shows that $h_l(0,r)$ is different from zero at $r=\infty$ so that $\varphi_l(0,r)$ is not normalizable and $k=0$ is not a discrete eigenvalue. For $l>0$, on the other hand, the inequality (3.15) shows that as $r\to\infty$,

$$h_l(0,r)=O(r^{-l});$$

$\varphi_l(0,r)$ is therefore square integrable and zero is a discrete eigenvalue if $f_l(0)=0$. We then have a zero energy-bound state. For $l=0$ this can happen only if the potential fails to satisfy (3.1); see Sec. 10f for an example.

The next question is naturally the multiplicity of the zeros of $f_l(k)$. Take first the case for which $f_l(k_0)=0$ with $\mathrm{Im}k_0<0$. Differentiation of (4.3) with respect to k (indicated by a dot), subsequently setting $k=k_0$ and using (4.18) leads to

$$\dot{f}_l(k_0)=k_0^lc^{-1}W[\dot{f}_l(k_0,r),f_l(k_0,r)] +k_0^lcW[\varphi_l(k_0,r),\dot{\varphi}_l(k_0,r)]. \tag{4.21}$$

The right-hand side can be evaluated by differentiating (4.19) and the equivalent equation for $f_l(k,r)$ with respect to k, then setting $k=k_0$. The result is that if $f_l(k_0)=0$, then

$$\dot{f}_l(k_0)=-2k_0^{l+1}\left[c\int_0^r dr'\varphi_l^2(k_0,r')+c^{-1}\int_r^\infty dr'f_l^2(k_0,r')\right]$$

$$=-2ck_0^{l+1}\int_0^\infty dr\varphi_l^2(k_0,r). \tag{4.21′}$$

Because of the boundary condition (3.4), $c\neq0$; furthermore, $\varphi_l(k_0,r)$ is real for purely imaginary k_0; hence the right-hand side of (4.21′) cannot vanish. Consequently, $\dot{f}_l(k_0)\neq0$ when $f_l(k_0)=0$ and the zero is *always simple.*

The point $k=0$ requires special consideration. Suppose that $f_l(0)=0$. We then take k first in the lower half-plane, $\mathrm{Im}k<0$, and differentiate the equivalent of (4.19) for $h_l(k,r)$ with respect to k, subsequently setting $k'=k$:

$$W[\dot{h}_l(k,r),h_l(k,r)]=-2k\int_r^\infty dr'h_l^2(k,r'). \tag{4.22}$$

The next step is to let k tend to naught inside a cone of opening angle less than π:

$$-\mathrm{Im}k\geq\epsilon|k|, \quad \epsilon>0.$$

It is then easily shown by means of the inequalities (3.16), (3.15), and (3.9) that the right-hand side of

(4.22) has the same limit with $k\to 0$ as

$$\lim_{k\to 0} 2k\int_r^\infty dr' w_l^2(kr')k^{2l} = 2\lim\left[k^{2l}\int_{kr}^{a} dz w_l^2(z) + k^{2l+1}\int_{a/k}^\infty dr' e^{-2ikr'} i^{2l}\right] = \begin{cases} 0, & \text{if } l\geq 1,\\ -i, & \text{if } l=0.\end{cases}$$

As a result, (4.22) leads to

$$W[\dot{h}_l(0,r),h_l(0,r)] = \begin{cases} 0, & \text{if } l\geq 1,\\ i, & \text{if } l=0,\end{cases} \quad (4.23)$$

when $k=0$ is approached as indicated. Equation (4.21) for $k_0=0$ reads

$$\dot{f}_l(0) = c^{-1}W[\dot{h}_l(0,r),h_l(0,r)] + cW[\varphi_l(0,r),\dot{\varphi}_l(0,r)], \quad (4.21'')$$

where we may let r tend to zero. Equation (4.23) thus shows that if $f_l(0)=0$, then

$$\dot{f}_l(0) = \begin{cases} ic^{-1}, & \text{if } l=0,\\ 0, & \text{if } l\geq 1.\end{cases} \quad (4.24)$$

For $l\geq 1$ one further differentiation is required. Since then $\dot{f}_l(0)=0$, we have from (4.21) and (4.22)

$$\ddot{f}_l(0) = \lim_{k\to 0} k^{-1}\dot{f}_l(k) = -2c\int_0^\infty dr\varphi_l^2(0,r). \quad (4.21''')$$

We therefore find that if $f_l(0)=0$, then as $k\to 0$ with $-\text{Im}k\geq \epsilon|k|$, $\epsilon>0$,

$$1/f_l(k) = \begin{cases} O(k^{-1}), & \text{if } l=0,\\ O(k^{-2}), & \text{if } l\geq 1,\end{cases} \quad (4.25)$$

as well as

$$f_l(k) = \begin{cases} O(k), & \text{if } l=0,\\ O(k^2), & \text{if } l\geq 1,\end{cases} \quad (4.25')$$

which is to say that f_l goes to zero *exactly* as k or k^2, respectively. If $f_l(k)$ is analytic in a neighborhood of $k=0$, then the statement is simply that, if $f_l(0)=0$, then the zero is simple for $l=0$ and double for $l\geq 1$.

We may now draw a conclusion concerning the number of zeros of $f_l(k)$ in the lower half of the complex plane. The function $f_l(k)$ being regular analytic there, its zeros cannot have a point of accumulation except possibly at $k=0$ or $k=\infty$. These two points cannot be accumulation points of roots either, the former because of (4.25) and the latter because of (4.16). Consequently the number of zeros must be finite. This proves that the number of discrete eigenvalues (i.e., bound states) for a given l value must be finite if the potential satisfies (3.1). An absolute bound on the number n_l of bound states of angular momentum l was given by Bargmann[22]:

$$n_l < \int_0^\infty drr|V(r)|/(2l+1), \quad (4.26)$$

which shows at the same time that the *total* number of bound states is finite.

In general we cannot say anything about the zeros of $f_l(k)$ in the upper half of the complex plane. They do not indicate eigenvalues. If $f_l(k_0)=0$ and both k_0 and $-k_0^*$ are in a region of analyticity of $f_l(k)$ connected with the real axis, then (4.7′) shows that we must also have $f_l(-k_0^*)=0$. The roots then appear in pairs symmetric with respect to the imaginary axis. Furthermore, Eq. (4.1) shows that if $k_0^2=-\kappa^2$, $\kappa>0$, is a discrete eigenvalue so that $f_l(-i\kappa)=0$, and if $+i\kappa$ lies in a region of analyticity of $f_l(k,r)$ connected with the real axis then we cannot have $f_l(i\kappa)=0$. Particularly, if the potential satisfies (3.14), then $f_l(i\kappa)$ cannot vanish if $-\kappa^2$ is an eigenvalue with $\kappa<a$. In other words, $f_l(i\kappa)$ can vanish under these circumstances only at the expense of a singularity of $f_l(k,r)$ at $k=i\kappa$.

If the potential vanishes identically beyond a finite distance R then quite a bit can be said about the zeros of $f_l(k)$. First of all, $f_l(k)$ must then have infinitely many complex roots in the upper half-plane. That fact is shown as follows.[23] The function

$$g_l(k^2) \equiv f_l(k)f_l(-k) \quad (4.27)$$

is in that case an entire function of k^2. Because of (2.22), (3.13), and (4.4) the asymptotic behavior of $f_l(k)$ when $\text{Im}k\to +\infty$ is

$$f_l(k) \sim -(-)^l(2ik)^{-1}e^{-2ikR}\int_0^R drV(r)e^{-2ik(r-R)}, \quad (4.28)$$

and hence by (4.16) that of $g_l(k^2)$ is the same. Suppose then that near $r=R$ the potential has an asymptotic expansion whose first term is

$$V(r) \sim c(R-r)^\sigma, \quad \sigma\geq 0. \quad (4.29)$$

Then (4.28) and (4.16) imply that as $\text{Im}k\to +\infty$

$$g_l(k^2) \sim \text{const.}\times k^{-\sigma-2}e^{-2ikR}. \quad (4.28')$$

Therefore, the *order*[27] of $g_l(k^2)$ is $\frac{1}{2}$. But an entire function of nonintegral order has necessarily an infinite number of zeros.[29] Because of (4.28′), moreover, and

[22] V. Bargmann, Proc. Natl. Acad. Sci. U. S. **38**, 961 (1952).

[23] This was shown first by Humblet,[24] then independently by Rollnik.[25] The more general proof below follows Regge.[26]

[24] J. Humblet, Mém. Soc. roy. sci. Liège, 4, 12 (1952).

[25] H. Rollnik, Z. Physik **145**, 639 and 654 (1956).

[26] T. Regge, Nuovo Cimento **8**, 671 (1958).

[27] The definition of the order ρ of an entire function is

$$\rho \equiv \lim_{r\to\infty}\sup(\log\log M(r)/\log r),$$

where $M(r)$ is the maximum modulus of the function for $|z|=r$, see Boas,[28] p. 8.

[28] R. P. Boas, *Entire Functions* (Academic Press, Inc., New York, 1954).

[29] See Boas,[28] p. 24.

the analyticity of g_l, only a finite number of them can lie on the imaginary axis. As a result, $g_l(k^2)$ has infinitely many *complex* roots, which appear symmetrically with respect to the real axis and with respect to the imaginary axis. Those in the upper half-plane must be roots of $f_l(k)$ and those in the lower, of $f_l(-k)$.

The same argument which excludes infinitely many zeros on the imaginary axis also excludes infinitely many zeros above any ray through the origin, since there also the right-hand side of (4.28′) has no zeros. At the same time it follows from (4.15) that $f_l(k)\to 1$ as $|k|\to\infty$ on any line parallel to the real axis in the upper half plane; consequently, the number of zeros in any strip above the real axis is finite. In other words, although the total number of roots is infinite, for any given positive numbers μ and ν there is but a finite number of them with imaginary part less than ν or with a ratio of imaginary to real part greater than μ.

Since it follows from (4.16) that

$$\int_1^\infty dk k^{-2}\log g_l(k^2)<\infty,$$

we can also immediately draw the conclusion that if $\{k_n\}$ are the roots of $f_l(k)$, then[30]

$$\sum_n|\mathrm{Im}(k_n^{-1})|<\infty.$$

Since the roots of $f_l(k)$ appear in pairs symmetric with respect to the imaginary axis we can also say that

$$\sum k_n^{-1} \text{ converges}$$

provided that we always add k_n^{-1} and $(-k_n^*)^{-1}$ together first.

The distribution of zeros k_n in the right half-plane can be shown[31] in more detail to be such that as $n\to\infty$

$$\begin{aligned}\mathrm{Re}k_n&=n\pi/R+O(1),\\ \mathrm{Im}k_n&=[(\sigma+2)/2R]\log n+O(1).\end{aligned}\tag{4.30}$$

The entire function $f_l(k)$ can now be written in the form of an infinite product. (We are still dealing with the case in which $V=0$ for $r>R$.) According to Hadamard's factorization theorem[32] we can write

$$f_l(k)=f_l(0)e^{-ick}\prod_1^\infty\left(1-\frac{k}{k_n}\right)e^{k/k_n}\tag{4.31}$$

assuming for simplicity $f_l(0)\neq 0$.[33] The constant c can be evaluated by means of a theorem by Pfluger[34] which tells us that the asymptotic behavior for large $|k|$ of (4.31) is for $k=\pm i|k|$

$$|k|^{-1}\log|f_l(k)/f_l(0)|=A\mp\sum \mathrm{Im}k_n^{-1}\pm c+o(1).$$

Comparison with (4.16) and (4.28′) shows that the right-hand side must equal $2R$ for $k=i|k|$, and zero, for $k=-i|k|$; hence, we must have

$$c+i\sum k_n^{-1}=R,$$

and consequently[26]

$$f_l(k)=f_l(0)e^{-ikR}\prod_1^\infty(1-k/k_n).\tag{4.31′}$$

If we differentiate the logarithm of this equation and set $k=0$ we obtain at once by (4.9)

$$R+\frac{d}{dk}\delta_l(k)\bigg|_{k=0}=\sum \mathrm{Im}k_n/|k_n|^2.\tag{4.32}$$

The only negative contributions on the right-hand side come from the bound states.

It should be reemphasized that all of the foregoing detailed conclusions are true only if the potential vanishes identically beyond a finite point.

We may now compare φ_l to the physical wave function ψ_l. Equation (4.8) shows the difference in their analytic properties. Under the hypotheses (3.1) $\psi_l(k,r)$ is in general regular only in the upper half of the complex k plane. But even there it has simple poles at $k=i|k_0|$ if k_0^2 is an eigenvalue. That is the reason why, in contrast to $\varphi_l(k,r)$, the physical wave function $\psi_l(k,r)$ cannot always be expanded in a Born series.

If we think of the potential multiplied by a possibly complex scale factor λ,

$$V\to\lambda V,\tag{4.33}$$

then we saw that $\varphi_l(k,r)$ can always be expanded in a power series in λ which converges absolutely for all values of λ, and so can $f_l(k)$. If, however, for a given value of k, $f_l(-k)=0$ when λ has some complex value λ_0, then the power series in λ for $\psi_l(k,r)$ (Born series) will certainly not converge absolutely for $\lambda\geq\lambda_0$. Thus the Born series for ψ_l will have a finite radius of convergence.

The inequality (4.15) shows directly an important fact about the Born series. Since for every λ, $f_l(k)$ differs arbitrarily little from unity when k is made large enough (real or in the lower half plane) $f_l(k)$ can, for any given complex λ, have no zeros on the real axis beyond a certain point. Hence for every potential that satisfies (3.1), the Born series for $\psi_l(k,r)$ will necessarily converge absolutely if only k is large enough. Furthermore, if k is sufficiently large, the first Born approximation is good.

As a function of E, ψ_l has a branch cut along the positive real axis. On the "physical" sheet ($\mathrm{Im}k\geq 0$) of its Riemann surface ψ_l is a regular analytic function of E, except for simple poles at the bound state energies $E=-|E_n|$. At $E=0$ it is at worst $O(E^{\frac{1}{2}})$ (when approached inside the first sheet of the Riemann surface).

[30] See Boas,[28] p. 134; the argument is due to Regge.[26]

[31] See Humblet,[24] p. 45; also Regge,[26] which contains an error of a factor of π in the denominator of Eq. (19).

[32] See, for example, Boas,[28] p. 22.

[33] Otherwise we must replace $f_l(0)$ by const$\times k$ for $l=0$, or by const$\times k^2$ for $l\geq 1$.

[34] A. Pfluger, Comm. Math. Helv. **16**, 1 (1943); theorem 6B.

5. PROPERTIES OF THE S MATRIX

We may now use (4.5) in order to draw conclusions concerning the S matrix from the properties of $f_l(k)$.

On the real axis S_l is continuous and because of (4.16)

$$\lim_{k\to\pm\infty} S_l(k)=1. \tag{5.1}$$

For any potential that satisfies (3.1) the phaseshift thus necessarily approaches an integral multiple of π at high energies. If, in addition, the potential is integrable at $r=0$ then (4.16′) yields immediately the Born approximation result as $k\to\pm\infty$

$$k\tan\delta_l(k)=-\frac{1}{2}\int_0^\infty drV(r)+o(1). \tag{5.1′}$$

If all we know about the potential is (3.1) then we can say nothing about the properties of $S_l(k)$ in the complex plane, since as soon as we leave the real axis either $f_l(k)$ or $f_l(-k)$ may fail to be regular. In other words, $S_l(k)$ may have singularities of any type anywhere in the complex plane off the real axis. We cannot even conclude from (4.5) that $S_l(k)$ has a pole at $k=i|k_0|$ is k_0^2 is a discrete eigenvalue, because $f_l(i|k_0|)$ may be zero.

If, however, the potential satisfies (3.14) then $S_l(k)$ is necessarily an analytic function regular in the strip $0\leq \mathrm{Im}k<a$, except for simple poles at $k=i\kappa_n$ whenever $-\kappa_n^2$ is an eigenvalue and $0<\kappa_n<a$. In specific cases the analytic continuation of $S_l(k)$ may be carried further than $|\mathrm{Im}k|=a$ by the argument following (4.7′); namely, the continuation of successive terms in the Born approximation.

If the potential decreases asymptotically more rapidly than every exponential, particularly if it vanishes identically outside a finite region, then S_l is regular in the entire upper half-plane except for simple poles at $k=i\kappa_n$, $\kappa_n>0$, whenever $-\kappa_n^2$ is a discrete eigenvalue. The residues at such poles are readily found[35] by (4.20), (4.21′), and (4.8):

$$\begin{aligned}\mathrm{Res}_n&=1/\dot{S}_l(-i\kappa_n)\\&=\frac{(-)^{l+1}i[f_l(i\kappa_n)]^2}{4\kappa_n^{2l+2}\int_0^\infty dr[\varphi_l(-i\kappa_n,r)]^2}\\&=i\bigg/4\int_0^\infty dr[\psi_l(-i\kappa_n,r)]^2,\end{aligned} \tag{5.2}$$

which is purely imaginary and

$$-i(-)^l\,\mathrm{Res}_n>0.$$

In the lower half of the complex plane singularities may again occur anywhere. If the potential fulfills (3.14) then $S_l(k)$ is regular there for $-\mathrm{Im}k<a$, except at isolated points where it may have poles of finite order. The latter occur at the zeros of $f_l(-k)$. For a potential of finite range that statement holds for the entire lower half-plane.

For a potential of type (3.14) then the zeros of $f_l(k)$ in the upper half-plane sufficiently close to the real axis lead to resonancelike peaks in $S_l(k)$ on the real axis.[38] The zeros of $f_l(k)$ on the positive imaginary axis are sometimes referred to as "virtual bound states." For a potential of finite range R we may immediately refer to the detailed discussion in Sec. 4 of the distribution of zeros of $f_l(k)$ in the upper half-plane. Thus there is always at most a finite number of virtual states and an infinite number of "resonances" distributed as shown in (4.30).

It is worthwhile to translate some of the foregoing statements into the language of *energy*. $S_l(E)$ then has a branch line along the positive real axis. If the potential satisfies (3.14) then $S_l(E)$ is an analytic function on a two sheeted Riemann surface, regular on the "physical sheet" ($\mathrm{Im}k\geq 0$) for $|E|<\hbar^2a^2/2\mu$, except for simple poles at $E=-|E_n|$, where E_n are the energies of bound states; on the sheet reached via the cut along the positive real axis $S_l(E)$ is regular for $|E|<\hbar^2a^2/2\mu$, except at a number of discrete points where it may have poles of finite order. The latter, if sufficiently close to the positive real axis, lead to resonancelike peaks in the functional behavior of $S_l(E)$ for positive E.

If the potential vanishes identically for $r>R$ then the foregoing statements hold on the entire Riemann surface (except at infinity); furthermore, on the first sheet we then have by (4.16) and (4.17)

$$\lim_{|E|\to\infty}[S_l(E)-1]e^{2ikR}=0. \tag{5.3}$$

If in addition the potential is absolutely integrable at the origin, then

$$[S_l(E)-1]|k|e^{2ikR}=O(1)\quad\text{as}\quad |E|\to\infty. \tag{5.3′}$$

In either case one may apply Cauchy's theorem to the integral

$$\int\frac{dE'[S_l(E')-1]e^{2ik'R}}{E'-E}$$

over a contour running above and below the branch cut and closed by a circle of large radius on the first sheet of the Riemann surface. Since the values of S_l on the upper and lower rim of the cut are related according to (4.5)–(4.7) by

$$S_l(E+i\epsilon)=S_l^*(E-i\epsilon),$$

[35] This result was first written down explicitly for $l=0$ by Lüders[36]; see also Hu.[37]

[36] G. Lüders, Z. Naturforsch. **10a**, 581 (1955).

[37] N. Hu, Phys. Rev. 74, 131 (1948).

[38] I should prefer not to refer to these peaks as resonances but to reserve that name for peaks which are indeed caused by a physical resonance phenomenon. Otherwise the term loses its physical content. In that sense, then, a single channel problem never has resonances except for the low energy type associated with a bound or "almost bound" virtual state. For the same reason I should not like to refer to the complex zeros of $f_l(k)$ as decaying or radioactive states.

the result is the dispersion relation,[39]

$$\mathrm{Re}[(S_l(E)-1)e^{2ikR}]=\sum_n \frac{(E_n/\kappa_n)\,\mathrm{Res}_n \exp(-2\kappa_n R)}{E-E_n} +\frac{1}{\pi}P\int_0^\infty dE' \frac{\mathrm{Im}[(S_l(E')-1)e^{2ik'R}]}{E'-E}, \quad (5.4)$$

where $E_n=-\hbar^2\kappa_n^2/2\mu$ are the bound state energies, $\{\mathrm{Res}_n\}$ are the residues of S_l at $k=i\kappa_n$ given by (5.2), and P denotes the Cauchy principal value. Because of the presence of e^{2ikR} this is not a very useful equation.

More practically, we may represent the S matrix by the use of (4.31′) as an infinite product[40]:

$$S_l(k)=e^{-2ikR}\prod_{n=1}^\infty \frac{k_n-k}{k_n+k}. \quad (5.5)$$

This amounts to writing the phase shift as

$$\delta_l(k)=-kR-\sum_{n=1}^N \tan^{-1}\frac{k}{\kappa_n} +\sum_{n=1}^{N'}\tan^{-1}\frac{k}{\kappa_n'}+\sum_{\alpha=1}^\infty \tan^{-1}\frac{2kk_\alpha^{(2)}}{|k_\alpha|^2-k^2}, \quad (5.5')$$

where $E_n=-\hbar^2\kappa_n^2/2\mu$ are the bound state energies, $E_n'=-\hbar^2\kappa_n'^2/2\mu$ are the virtual state energies, and $E_\alpha=\hbar^2(k_\alpha^{(1)}+ik_\alpha^{(2)})^2/2\mu$, with $k_\alpha^{(1)}>0$, $k_\alpha^{(2)}>0$, are the "resonances." The distribution of the energies E_α is such that for any two given positive numbers c and d there exists only a finite number of E_α's above the ray $\mathrm{Im}E_\alpha=c\,\mathrm{Re}E_\alpha$ or below the parabola $\mathrm{Im}E_\alpha=2d\times(d^2+\mathrm{Re}E_\alpha)^{\frac12}$.

Notice that each term in the α-sum contributes an increase in the phaseshift by π in the vicinity of $k\approx|k_\alpha|$. The contribution of the linear decrease in the first term, however, is such that almost all these increases are compensated by subsequent decreases. If the rising part of the curve leads to a "resonance" (i.e., a value of δ_l which is an odd multiple of $\frac12\pi$) then so must the falling part (for almost all α), although that type of "resonance" bears no relation to the k_α.

Another point to notice is that, in spite of the appearance of infinitely many "resonance" terms in (5.5′), because of (5.1) only a finite number of them can actually lead to $\sin^2\delta_l=1$. There always exists an energy beyond which this can no longer occur.[41] Furthermore, it is easily seen by means of (4.30) that the maximal slope of individual terms in the α sum, which for large α occurs at $k\sim k_\alpha^{(1)}$, tends to naught as $1/k_\alpha^{(2)}$. Consequently, not only are there but a finite number of "resonance" points, but beyond a certain energy the phase shift becomes monotonely decreasing.

Another way of representing $S_l(k)$ if $V(r)=0$ for $r>R$ is a Mittag-Leffler expansion.[42] In order to do that we need an estimate for the residues of $S_l(k)$ at $k=-k_n$ if $f_l(k_n)=0$. Because of the distribution of zeros given by (4.30) and by (4.4) the leading terms of $f_l(k)$ in the vicinity of k_n when $n\to\infty$ are [assuming integrability of $V(r)$]

$$f_l(k)=1-(-)^l(2ik)^{-1}\int_0^R dr V(r)e^{-2ikr}+O(k_n^{-1}), \quad (5.6)$$

while

$$\dot f_l(k)=(-)^l k^{-1}\int_0^\infty dr\, r V(r)e^{-2ikr}+O(k_n^{-1}).$$

If we assume (4.29) then it is readily seen that we get

$$\dot f_l(k)=-2iR[f_l(k)-1]+O(k_n^{-1})$$

since for large k only the vicinity of $r=R$ contributes to the integral. If we now evaluate $\dot f_l$ at $k=k_n$ then we obtain,

$$\dot f_l(k_n)=2iR+O(k_n^{-1}). \quad (5.7)$$

Consequently, the residue of $S_l(k)$ at $k=-k_n$,

$$R_n=-f_l(-k_n)/\dot f_l(k_n)$$

is by (4.16′),

$$R_n=i/2R+O(k_n^{-1}). \quad (5.8)$$

As a result of (5.8), (4.6) and its unitarity, the S matrix can be written[24]

$$S_l(k)=1+kP_l(k)-k\sum_n\left(\frac{R_n/k_n}{k+k_n}+\frac{R_n^*/k_n^*}{k-k_n^*}\right), \quad (5.9)$$

where $P_l(k)$ is an entire function of k and it is understood that $\mathrm{Re}k_n\geq 0$, and that for the finite number of purely imaginary poles of $S_l(k)$, R_n is *one-half* the (purely imaginary) residue. Since by (5.8) and (4.30) for large n

$$\frac{R_n}{k_n^2}+\frac{R_n^*}{k_n^{*2}}=iR(\pi n)^{-2}+O(n^{-4}\log^2 n),$$

the series in (5.9) converges for all k and the Mittag-Leffler expansion is established.

As a consequence of (5.9) we can write

$$\mathrm{Re}[1-S_l(k)]=2\sin^2\delta_l(k) =EQ_l(E)+E\sum_n \frac{A_n(E-E_n)+\frac12 B_n\Gamma_n}{(E-E_n)^2+\frac14\Gamma_n^2}, \quad (5.9')$$

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[39] See, e.g., E. Corinaldesi, Nuclear Phys. 2, 420 (1956).

[40] Such a product representation was written down by Hu,[37] for example, but not proved. It was proved under the conditions of this paper by Regge.[26]

[41] It is easily seen by considering $h_l(k)\equiv f_l(k)+f_l(-k)$ that whenever the potential is at least exponentially decreasing at infinity there can be only a finite number of points where $\sin^2\delta_l=1$, i.e., $h_l(k)=0$. In that case $h_l(k)$ is analytic on the real axis and, by (4.16), approaches unity at $k\to\pm\infty$; hence it cannot have infinitely many real zeros. If the potential does not satisfy (3.14) for any positive value of a then no such conclusion can be drawn.

[42] See, for instance, C. Caratheodory, *Functionentheory I* (Birkhäuser, Basel, 1950), p. 215 ff. This was first proved by Humblet[24] although written down without proof before, for example, by Hu.[37] The argument below is a simplified and somewhat less rigorous version of Humblet.[24]

where

$$A_n+iB_n=2R_n/k_n,$$

$$E_n+\tfrac{1}{2}i\Gamma_n=\hbar^2k_n^2/2\mu,$$

and

$$Q_l(E)=\tfrac{1}{2}[P_l(-k)-P_l(k)]k/E$$

is an entire function of E. The residues R_n can be expressed in terms of the k_n. By (5.5) we have

$$\frac{R_n}{k_n}=2e^{2ik_nR}\prod_{m\neq n}\frac{k_m+k_n}{k_m-k_n}. \quad (5.10)$$

The entire function $P_l(k)$ is also determined by the k_n; but no simple expression is known.

An interesting relation between the phase shift and the wave function, when the potential vanishes for $r>R$, is obtained as follows:

Differentiating the equivalent of (4.19) for $f_l(k,r)$ and $\varphi_l(k',r)$ with respect to k' and then setting $k=k'$ yields after integration

$$W[f_l(k,r),\dot{\varphi}_l(k,r)]=-2k\int_0^r dr' f_l(k,r')\varphi_l(k,r')$$

since $\dot{\varphi}_l'(k,r)=o(r^l)$ and $\dot{\varphi}_l(k,r)=o(r^{l+1})$ as $r\rightarrow 0$. Differentiation of (4.3) with respect to k therefore yields

$$\dot{f}_l(k)=lk^{-1}f_l(k)-2k^{l+1}\int_0^r dr' f_l(k,r')\varphi_l(k,r') + k^lW[\dot{f}_l(k,r),\varphi_l(k,r)],$$

where $r\geq R$. We then find by (4.1) that

$$\frac{\dot{f}_l(k)}{f_l(k)}-\frac{\dot{f}_l(-k)}{f_l(-k)}=\frac{d}{dk}\log S_l(k)$$
$$=\frac{2ik^{2l}}{|f_l(k)|^2}\Big\{2k^2\int_0^r dr'\varphi_l^2(k,r')$$
$$-r[\varphi_l'^2(k,r)-\varphi_l(k,r)\varphi_l''(k,r)]+\varphi_l(k,r)\varphi_l'(k,r)\Big\},$$

or by (4.8), for $r\geq R$,

$$\frac{d}{dk}\delta_l(k)=2\int_0^r dr'|\psi_l(k,r')|^2$$
$$-k^{-2}\{r[|\psi_l'(k,r)|^2-\psi_l''(k,r)\psi_l^*(k,r)]$$
$$-\psi_l'(k,r)\psi_l^*(k,r)\}. \quad (5.11)$$

This equation takes on its most transparent form for $l=0$. Since it follows from (5.11) that

$$2|\psi_l(k,r)|^2=k^{-2}\frac{d}{dr}\{\ \},$$

we may also write

$$\frac{d}{dk}\delta_0(k)=2\int_0^\infty dr[|\psi_0(k,r)|^2-|\psi_0^{\text{out}}(k,r)|^2] + \tfrac{1}{2}k^{-1}\sin 2\delta_0(k), \quad (5.11')$$

where ψ_l^{out} is the free wave function equal to ψ_l for $r\geq R$ and then continued in for $r<R$. For $l\neq 0$ this cannot be done since ψ_l^{out} is then not square integrable at the origin. Equation (5.11′) directly illuminates the significance of a "resonance." Whenever the phaseshift varies rapidly upwards it means that there is a large probability for the particles to be found inside the region of interaction.

We may also write ψ_0^{out} explicitly; thus[43]

$$\frac{d}{dk}\delta_0(k)=2\int_0^R dr|\psi_0(k,r)|^2-R + \tfrac{1}{2}k^{-1}\sin(2kr+2\delta_0). \quad (5.11'')$$

It follows from this that

$$(d/dk)\delta_0(k)>-R+\tfrac{1}{2}k^{-1}\sin(2kr+2\delta_0)$$
$$\geq -(R+\tfrac{1}{2}k^{-1}). \quad (5.12)$$

We may also compare Eq. (5.11) with (5.5); that leads to

$$2\int_0^\infty dr[|\psi_0(k,r)|^2-|\psi_0^{\text{out}}(k,r)|^2]+\tfrac{1}{2}k^{-1}\sin 2\delta_0(k)$$
$$=\sum_n\frac{ik_n}{k_n^2-k^2}-R. \quad (5.13)$$

It should be recalled at this point that all the results from Eq. (5.3) on assumed that the potential vanishes identically for $r>R$. We now return to the general case, assuming only (3.1).

The low-energy behavior of $S_l(k)$ is established as follows: The analytic function $f_l(k)$ being regular in the lower half of the complex k plane, we have

$$\frac{1}{2\pi i}\int_C d\log f_l(k)=n_l, \quad (5.14)$$

where n_l is the number of zeros of $f_l(k)$ in the lower half-plane and the path of integration C runs along the real axis from $+\infty$ to $-\infty$, avoiding the origin by a small semicircle of radius ϵ in the lower half-plane, and closed by a large semicircle of radius K in the lower half-plane. Since each discrete eigenvalue produces a simple zero of $f_l(k)$, n_l is the number of bound states of angular momentum l.

The contribution to (5.14) from the large semicircle vanishes by (4.16) in the limit as $K\rightarrow\infty$. If near $k=0$ we write[44]

$$f_l(k)=ak^q+o(k^q)$$

[43] This equation and the following inequality were given by Lüders.[36] The inequality (5.12) and the corresponding one obtainable from (5.11) for $l=1$ were first derived by Wigner under more general assumptions; E. P. Wigner, Phys. Rev. **98**, 145 (1955).

[44] It is sufficient that that is true in every cone of opening less than π in the lower half-plane.

then the contribution from the small semicircle is

$$\int_{\circlearrowleft} d\log f_l(k) \to q\int_{\circlearrowleft} d\log k = -i\pi q$$

in the limit as $\epsilon\to 0$. Consequently, (5.14) becomes, by (4.9),

$$\lim_{\epsilon\to 0}\lim_{K\to\infty}\Big[\delta_l(-K)-\delta_l(-\epsilon)+\delta_l(\epsilon)-\delta_l(K) +i\log\left|\frac{f_l(K)}{f_l(-K)}\cdot\frac{f_l(-\epsilon)}{f_l(\epsilon)}\right|\Big]=2\pi(n_l+\tfrac{1}{2}q).$$

The imaginary term vanishes by (4.7). Furthermore, because of (4.7), $\delta_l(k)$ may be defined to be an odd function of k. We have therefore

$$\delta_l(0)-\delta_l(\infty)=\pi(n_l+\tfrac{1}{2}q).$$

A glance at (4.25) and (4.25′) shows that $q=0$ if $f_l(0)\neq 0$; $q=1$ if $f_l(0)=0$ and $l=0$; $q=2$ if $f_l(0)=0$ and $l\geq 1$. In the last case, as we saw, $k=0$ is a discrete eigenvalue and should thus be added to n_l. As a result we obtain the Levinson theorem[45]

$$\delta_l(0)-\delta_l(\infty) =\begin{cases}\pi(n_l+\frac{1}{2}), & \text{if } l=0 \text{ and } f_l(0)=0,\\ \pi n_l, & \text{otherwise,}\end{cases} \qquad (5.15)$$

which constitutes the only generally valid relation between scattering phaseshifts and bound states.

Because of (4.16), $\delta_l(\infty)$ may always be defined to be zero. Equation (5.15) then determines the value of the phaseshift at zero energy. As a consequence of (5.15) we have

$$S_l(0)=\begin{cases}-1, & \text{if } l=0 \text{ and } f_l(0)=0,\\ 1, & \text{otherwise.}\end{cases} \qquad (5.16)$$

Notice that $S_l(0)=-1$ implies by (2.15) that the scattering amplitude becomes infinite at $E=0$. This happens whenever the potential is such that the slightest strengthening will introduce a new bound state of zero angular momentum. This is usually referred to as a zero-energy resonance.

The next question that arises is how $S_l(k)$ approaches its limiting value at $k=0$. The answer is given most simply by using (2.21) in combination with (4.8)

$$S_l(k)=1-2ik^l\int_0^\infty dr u_l(kr)V(r)\varphi_l(k,r)/f_l(-k). \qquad (5.17)$$

[45] Although this theorem was known before in less precise form, it was proved first by Levinson.[13] The proof given here follows Levinson. It has been proved under more general hypotheses, namely, only the completeness of the set of eigenfunctions by J. Jauch, Helv. Phys. Acta **30**, 143 (1957); see also A. Martin, Nuovo cimento **7**, 607 (1958).

The inequalities (3.9) and (3.11) show that

$$|S_l(k)-1|\leqslant C|k|^{2l+1}\int_0^\infty dr|V(r)| \times\left(\frac{r}{1+|k|r}\right)^{2l+2}\Big/|f_l(k)|. \qquad (5.18)$$

If $f_l(0)\neq 0$ then we may conclude that the right-hand side is $O(k^{2l+1})$ *provided* that[46]

$$\int_0^\infty dr|V(r)|r^{2l+2}<\infty.$$

If $f_l(0)=0$, then it follows from (4.25) that the right-hand side of (5.18) is $O(k^{2l-1})$ for $l\geq 1$, and $O(1)$ for $l=0$. Consequently, if the potential satisfies the foregoing restriction, then as $k\to 0$,

$$S_l(k)-1=2ie^{i\delta_l}\sin\delta_l =\begin{cases}-2, & \text{if } l=0 \text{ and } f_l(0)=0,\\ O(k^{2l+1}), & \text{otherwise.}\end{cases} \qquad (5.19)$$

If there is a bound state of zero energy (which is possible only if $l\geq 1$), then as $k\to 0$

$$S_l(k)-1=O(k^{2l-1}). \qquad (5.19')$$

We may generate a Born series for $S_l(k)$ by using (5.17). Just as in the case of $\psi_l(k,r)$, it is the presence of $f_l(-k)$ in the denominator which may prevent the convergence; as (5.18) shows, the numerator converges absolutely. We may, therefore, draw the same conclusion as at the end of Sec. 4. For every potential that fulfills (3.1) there exists an energy beyond which the S matrix can be expanded in an absolutely convergent Born series. Furthermore, it follows from (5.17) together with (4.16) and (3.13) that in the high-energy limit the first term in the Born series is a good approximation.

Suppose that the potential $V=-|U|$ produces neither a bound state of $l=0$ nor a "zero energy resonance." Then $f_0(0)\neq 0$; the replacement (4.33) with $|\lambda|\leq 1$ cannot make $f_0(0)=0$ either, for that would imply

$$-\lambda\int_0^\infty drV(r)|\varphi_0(0,r)|^2=-\int_0^\infty dr\varphi_0{}^*(0,r)\varphi_0''(0,r) =\int_0^\infty dr|\varphi_0'(0,r)|^2,$$

which is possible only for real λ. As a result the Born series for

$$\lim_{k\to 0}[S_0(k)-1]/k=\dot{f}_0(0)/f_0(0)$$

[46] D. S. Carter, Ph.D. thesis, Princeton University, 1952 (unpublished).

converges absolutely. A glance at (2.15) shows that because of (5.19) the Born series for the scattering amplitude at zero energy then converges absolutely. But insertion of (2.2) and of the middle form of (2.3) in (2.12) shows that the Born series for $\Theta(\mathbf{k}',\mathbf{k})$ using U is dominated by that for $\Theta(0,0)$ using $V=-|U|$. Consequently, a sufficient criterion for the Born series for the scattering amplitude of a potential U to converge absolutely at all energies is that $V=-|U|$ produce no s wave bound states or zero energy resonance.[47]

We finally want to look at a question of more restricted applicability: Is it possible to determine the Jost function $f_l(k)$ from the knowledge of $S_l(k)$? The answer is "yes," provided that we know also *in addition* the energies of the bound states. [Their number is already determined by $S_l(k)$ according to (5.15).]

If there are n_l bound states with energies $-\hbar^2\kappa_n^2/2\mu$, $\kappa_n\geq 0$, then we know that $S_l(k)$ can be written by (4.5) as

$$S_l(k)=\prod_{n=1}^{n_l}\left(\frac{k+i\kappa_n}{k-i\kappa_n}\right)^2 S_l^{\text{red}}(k),$$

where

$$S_l^{\text{red}}(k)=f_l^{\text{red}}(k)/f_l^{\text{red}}(-k),$$

and

$$f_l^{\text{red}}(k)=\prod\frac{k-i\kappa_n}{k+i\kappa_n}f_l(k) \tag{5.20}$$

is an analytic function regular in the lower half of the complex plane, without any zeros there, and with

$$f_l^{\text{red}}(k)\underset{|k|\to\infty}{\longrightarrow}1$$

there. Therefore $\log f_l^{\text{red}}(k)$ is analytic in the lower half-plane and vanishes at infinity; consequently, it satisfies a simple "dispersion relation." By Cauchy's theorem

$$\log f_l^{\text{red}}(k)=-\frac{P}{\pi i}\int_{-\infty}^{\infty}dk'\frac{\log f_l^{\text{red}}(k')}{k'-k},$$

and hence

$$\log|f_l(k)|=-\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{dk'\,\text{Im}\log f_l^{\text{red}}(k')}{k'-k}.$$

By (5.20) and (4.9) we have

$$\text{Im}\log f_l^{\text{red}}(k)=\delta_l(k)-2\textstyle\sum_n\cot^{-1}(k/\kappa_n)$$

and therefore[48]

$$\log|f_l(k)|=-\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{dk'\delta_l(k')}{k'-k}+\sum_n\log\frac{E-E_n}{E} \tag{5.21}$$

or

$$f_l(k)=\prod_n\left(1-\frac{E_n}{E}\right)\exp\left[-\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{dk'\delta_l(k')}{k'-k+i\epsilon}\right] \tag{5.21'}$$

in the limit as $\epsilon\to 0+$. This explicitly expresses $f_l(k)$ in terms of the phaseshift $\delta_l(k)$ (i.e., of $S_l(k)$) and the bound states.

6. GREEN'S FUNCTION

It is very easy now to write down a complete Green's function or resolvent of the radial Schrödinger equation (2.10). Such a function must satisfy the equation

$$\left[-\frac{d^2}{dr^2}+V(r)+\frac{l(l+1)}{r^2}-k^2\right]\mathfrak{G}_l(k;r,r')=-\delta(r-r'). \tag{6.1}$$

It is therefore a solution of (2.10) for $r\neq r'$. At $r=r'$ its derivative suffers a discontinuity of unity:

$$\frac{d}{dr}\mathfrak{G}_l(k;r,r')\Big|_{r=r'-\epsilon}^{r=r'+\epsilon}=1. \tag{6.2}$$

Suppose, then, we want the Green's function appropriate to the boundary condition of (2.8), namely, such that it contains no incoming waves at infinity. It must then have the form

$$\mathfrak{G}_l(k;r,r')=\begin{cases}\varphi_l(k,r)a(k,r'), & r<r',\\ f_l(k,r)b(k,r'), & r>r'.\end{cases}$$

Since $\mathfrak{G}_l$ must be continuous at $r=r'$ we find that

$$a(k,r)=C(k)f_l(-k,r),$$
$$b(k,r)=C(k)\varphi_l(k,r).$$

The requirement (6.2) then fixes $C(k)$

$$C(k)W[\varphi_l(k,r),f_l(-k,r)]=1.$$

Because of (4.3) we then find that

$$C(k)=(-)^{l+1}k^l/f_l(-k).$$

As a result

$$\begin{aligned}\mathfrak{G}_l(k;r,r')&=(-)^{l+1}k^l\varphi_l(k,r_<)f_l(-k,r_>)/f_l(-k)\\&=(-)^{l+1}k^{-1}\psi_l(k,r_<)f_l(-k,r_>).\end{aligned} \tag{6.3}$$

From the analyticity of $\varphi_l(k,r)$, $f_l(k,r)$ and $f_l(k)$ we can therefore infer the following properties of the Green's function $\mathfrak{G}_l(k;r,r')$.

For each fixed r and r', $\mathfrak{G}_l(k;r,r')$ is an analytic function of k regular in the open upper half of the complex plane and continuous on the real axis, except for simple poles at $k=i\kappa_n$ if $-\kappa_n^2$ is a discrete eigenvalue, and except at $k=0$ where it may be as singular as $O(k^{-2})$ when approached from above the real axis (for $l=0$, it is at worst $O(k^{-1})$).

In the language of the energy E, $\mathfrak{G}_l(k;r,r')$ has a branch cut along the positive real axis. On the "physical" sheet of its Riemann surface ($\text{Im}k\geq 0$) it is a regular analytic function except for a finite number of simple poles on the negative real axis at the position of the bound states. At $E=0$, it has a branch point and

[47] The foregoing argument is a slight generalization of that given by H. Davies, Nuclear Phys. 14, 465 (1960).

[48] The phaseshift is assumed to be defined so that it vanishes at infinity. A similar, but slightly less explicit form of $f_l(k)$ was given by Jost and Kohn.[49]

[49] R. Jost and W. Kohn, Phys. Rev. 87, 977 (1952).

may in exceptional cases be $O(E^{-1})$ when $E=0$ is approached from the first sheet of the Riemann surface. [For $l=0$, $O(E^{-\frac{1}{2}})$.] As $E\to\infty$ on the first sheet, it follows from (3.13), (3.17), and (3.16) that

$$\begin{aligned}\mathfrak{G}_l(E;r,r')&=G_l(E;r,r')+o(|k|^{-1}e^{-\nu|r-r'|})\\&=(2ik)^{-1}(e^{ik(r_>-r_<)}-(-)^l e^{ik(r_>+r_<)})\\&\qquad+o(|k|^{-1}e^{-\nu|r-r'|}).\end{aligned}\quad(6.4)$$

The function $\mathfrak{G}_l(E;r,r')$ taken at the upper rim of its branch cut (i.e., at $k>0$) is usually denoted by $\mathfrak{G}_l^{(+)}$; when we follow its analytic continuation around the origin to the lower rim of the cut ($k<0$) we obtain $\mathfrak{G}_l^{(-)}$.

In general nothing can be said about a possible analytic continuation of $\mathfrak{G}_l$ beyond the branch cut onto the second sheet. If the potential satisfies (3.14), then it is regular there as far as $|E|<a^2\hbar^2/2\mu$, except possibly for poles of finite order. The latter, if sufficiently close to the positive real axis, lead to resonancelike peaks in the scattering amplitude. If the potential vanishes at infinity faster than every exponential (e.g., if it is identically zero beyond a finite point), then $\mathfrak{G}_l$ has an analytic continuation into the whole second sheet of its Riemann surface, where it then must have infinitely many poles of finite order.

The relation between $\mathfrak{G}_l$ and S_l is given directly by the solution of (2.8):

$$\psi_l(k,r)=u_l(kr)+\int_0^\infty dr'\mathfrak{G}_l(k;r,r')V(r')u_l(kr'),\quad(6.5)$$

which, inserted in (2.21), yields

$$\begin{aligned}S_l(k)=1-2ik^{-1}&\int_0^\infty dr u_l(kr)V(r)u_l(kr)\\&-2ik^{-1}\int_0^\infty dr\int_0^\infty dr' u_l(kr)V(r)\\&\qquad\times\mathfrak{G}_l(k;r,r')V(r')u_l(kr').\end{aligned}\quad(6.6)$$

The difference in analytic behavior between $S_l(k)$ and $\mathfrak{G}_l(k;r,r')$, the latter being regular in the upper half-plane (except for the bound state poles), while the former need not be regular there, comes from the possible divergence of the integrals in (6.6).

7. COMPLETENESS

We now want to prove the completeness of the set of eigenfunctions of the radial Schrödinger equation under the assumption (3.1) on the potential. The idea of the proof[50] is to evaluate the integral

$$\int dE\mathfrak{G}_l(E;r,r')$$

over a closed contour running along the two rims of the branch cut in the complex E plane and closed by a large circle at infinity on the first sheet of the Riemann surface. On the one hand, that integral is evaluated by Cauchy's residue theorem in terms of the bound state poles on the negative real axis. On the other hand, it is explicitly written down in terms of its various contributions. The whole procedure is a little simpler, however, in the k plane.

We consider the integral

$$I(r)\equiv\int_C kdk\int_0^\infty dr'h(r')\mathfrak{G}_l(-k;r,r'),\quad(7.1)$$

where $\mathfrak{G}_l$ is given by (6.3), $h(r)$ is an arbitrary sufficiently well behaved function of r (square integrability suffices), and the contour C of the k integration is the same as in (5.14).

The integral $I(r)$ is written

$$I=I_1+I_2$$

$$I_1(r)=-\int_C dkk^{l+1}\int_0^r dr'h(r')\varphi_l(k,r')f_l(k,r)/f_l(k)\quad(7.2)$$

$$I_2(r)=-\int_C dkk^{l+1}\int_r^\infty dr'h(r')f_l(k,r')\varphi_l(k,r)/f_l(k).\quad(7.3)$$

We first consider $I_1(r)$.

Suppose that the discrete eigenvalues are $-\kappa_n^2$. Then we write ($\kappa_n>0$)

$$\varphi_l^{(n)}(r)\equiv\varphi_l(-i\kappa_n,r),$$
$$f_l^{(n)}(r)\equiv f_l(-i\kappa_n,r),$$
$$C_n\equiv\dot{f}_l(-i\kappa_n).$$

Since $\varphi_l(k,r)$, $f_l(k,r)$, and $f_l(k)$ are analytic functions regular in the lower half of the complex plane and $f_l(k)$ has simple zeros at $k=-i\kappa_n$, the integral $I_1(r)$ is evaluated immediately by means of Cauchy's residue theorem

$$I_1(r)=-2\pi i\sum_n\int_0^r dr'h(r')\varphi_l^{(n)}(r')f_l^{(n)}(r)(-i\kappa_n)^{l+1}/C_n.$$

If we call

$$\int_0^\infty dr[\varphi_l^{(n)}(r)]^2\equiv N_n^2,$$

then (4.21′) reads

$$C_n=-2a_n(-i\kappa_n)^{l+1}N_n^2,$$

where

$$a_n=f_l^{(n)}(r)/\varphi_l^{(n)}(r).$$

Thus we obtain

$$I_1(r)=i\pi\sum_n\int_0^r dr'h(r')\varphi_l^{(n)}(r')\varphi_l^{(n)}(r)N_n^{-2}.\quad(7.4)$$

[50] This proof follows Jost and Kohn,[49] Appendix. It is the type of proof given by Titchmarsh, see E. C. Titchmarsh, *Eigenfunction Expansions I* (Oxford University Press, New York, 1946).

On the other hand, we evaluate I_1 directly. The contribution $I_{1\epsilon}$ to the k integral from the small semicircle vanishes in the limit as its radius tends to zero, except when $f_l(0)=0$. In that case it still vanishes for $l=0$, because of (4.25). For $l\geq 1$ we write

$$f_l(k)=c_0k^2+o(k^2).$$

The contribution from the small semicircle is then by (4.21) seen to be

$$I_{1\epsilon}(r)=-i\pi\int_0^r dr'h(r')\varphi_l^{(0)}(r')\varphi_l^{(0)}(r)N_0^{-2}, \quad (7.5)$$

where

$$\varphi_l^{(0)}(r)\equiv\varphi_l(0,r),$$

and

$$N_0^2\equiv\int_0^\infty dr[\varphi_l^{(0)}(r)]^2.$$

The contribution I_{1R} to I_1 from the large semicircle is evaluated by the use of the asymptotic functions for large $|k|$. Thus by (3.13) and (3.17)

$$I_{1R}\underset{R\to\infty}{\sim}\tfrac{1}{2}i\int_0^r dr'h(r')\int_{s.c.}dk(e^{-ik(r-r')}-(-)^l e^{-ik(r+r')})$$

$$\underset{R\to\infty}{\sim}\tfrac{1}{2}h(r)\int_{s.c.}dkk^{-1}=\tfrac{1}{2}i\pi h(r). \quad (7.6)$$

The remaining contribution I_{1E} to I_1 is the integral over the real axis, where we may use the fact that $\varphi_l(k,r)$ is an even function of k and then (4.1) and (4.7); thus

$$I_{1E}(r)=\int_0^r dr'h(r')\left(\int_{-\infty}^{-\epsilon}+\int_{+\epsilon}^{\infty}\right)dkk^{l+1}\times\varphi_l(k,r')f_l(k,r)/f_l(k)$$

$$=-i\int_0^r dr'h(r')\left(\int_{-\infty}^{-\epsilon}+\int_{+\epsilon}^{\infty}\right)dkk^{2l+2}\times\varphi_l(k,r')\varphi_l(k,r)/|f_l(k)|^2.$$

We may now let $\epsilon\to 0$ and get

$$I_{1E}(r)=-2i\int_0^r dr'h(r')\int_0^\infty dkk^{2l+2}\times\varphi_l(k,r')\varphi_l(k,r)/|f_l(k)|^2. \quad (7.7)$$

Equating the sum of (7.5)–(7.7) to (7.4) yields

$$h(r)=2\int_0^r dr'h(r')\left[2\int_0^\infty dkk^{2l+2}\frac{\varphi_l(k,r')\varphi_l(k,r)}{\pi|f_l(k)|^2}+\sum_n\frac{\varphi_l^{(n)}(r')\varphi_l^{(n)}(r)}{N_n^2}\right], \quad (7.8)$$

where the sum now includes the bound state of zero binding energy if there is one.

We then go through the same arguments for $I_2(r)$, where we may replace the upper limit of the r' integration by $r+\mu$, μ being an arbitrary positive number. The result is

$$h(r)=2\int_r^{r+\mu} dr'h(r')\left[2\int_0^\infty dkk^{2l+2}\frac{\varphi_l(k,r')\varphi_l(k,r)}{\pi|f_l(k)|^2}+\sum_n\frac{\varphi_l^{(n)}(r')\varphi_l^{(n)}(r)}{N_n^2}\right]. \quad (7.8')$$

We now add (7.8) and (7.8′), divide by two, and let $\mu\to\infty$. The ensuing improper integral will converge provided $h(r)$ is square integrable. The result can be written in the customary notation of a δ function

$$\frac{2}{\pi}\int_0^\infty dkk^{2l+2}\frac{\varphi_l(k,r)\varphi_l(k,r')}{|f_l(k)|^2}+\sum_n\frac{\varphi_l^{(n)}(r)\varphi_l^{(n)}(r')}{N_n^2}=\delta(r-r'). \quad (7.9)$$

This proves the completeness of the set of wave functions of the continuous and discrete spectrum and shows at the same time what the necessary weight function is.

The weight function appearing in (7.9) is also defined as the *spectral function* $\rho_l(E)$ in the following sense. If we set

$$\frac{d\rho_l(E)}{dE}=\begin{cases}\dfrac{2\mu}{\pi\hbar^2}k^{2l+1}/|f_l(k)|^2, & E>0,\\ \sum_n \delta(E-E_n)/N_n^2, & E\leq 0,\end{cases} \quad (7.10)$$

with $\rho_l(-\infty)=0$, then (7.9) can be written as a Stieltjes integral

$$\int d\rho_l(E)\varphi_l(k,r)\varphi_l(k,r')=\delta(r-r'). \quad (7.9')$$

At the same time we may now write the resolvent (6.3)

$$\mathfrak{G}_l(E;r,r')=\frac{\hbar^2}{2\mu}\int d\rho_l(E')\frac{\varphi_l(k',r)\varphi_l(k',r')}{E-E'}. \quad (7.11)$$

On the upper rim of the branch cut we get the outgoing wave Green's function $\mathfrak{G}_l^{(+)}$; on the lower rim, the incoming wave Green's function $\mathfrak{G}_l^{(-)}$. The average of the two defines a real (standing wave) Green's function, $\mathfrak{G}_l^{(P)}$, for which the Cauchy principal value of the integral must be used.

A comparison of (7.9) with (4.8) shows that we may write the completeness in terms of the physical wave function ψ_l

$$\frac{2}{\pi}\int_0^\infty dk\psi_l(k,r)\psi_l^*(k,r')+\sum_n\psi_l^{(n)}(r)\psi_l^{(n)}(r')=\delta(r-r'), \quad (7.9'')$$

where $\psi_l^{(n)}(r)$ are the bound-state wave functions normalized to unity

$$\psi_l^{(n)}(r)=\varphi_l(-i\kappa_n, r)/N_n.$$

The complete Green's function can similarly be written

$$\mathfrak{G}_l(k;r,r')$$
$$=\frac{2}{\pi}\int_0^\infty dk'\frac{\psi_l(k',r)\psi_l^*(k',r')}{k^2-k'^2}+\sum_n\frac{\psi_l^{(n)}(r)\psi_l^{(n)}(r')}{k^2+\kappa_n^2}$$
$$=\frac{1}{\pi}\int_{-\infty}^\infty\frac{dk'}{k'}\frac{\psi_l(k',r)\psi_l^*(k',r')}{k-k'}+\sum_n\frac{\psi_l^{(n)}(r)\psi_l^{(n)}(r')}{k^2+\kappa_n^2}, \tag{7.11'}$$

where for real k the limit from above the real axis to positive k defines the outgoing wave Green's function, and to negative k, the incoming wave Green's function.

8. GEL'FAND-LEVITAN EQUATIONS

The equations first derived by Gel'fand and Levitan[51,52] have a special interest for the solution of the problem of going backwards, from a knowledge of the phaseshift and bound states to the underlying potential. However, they are useful sometimes also in other contexts.

Consider the function[53]

$$I(E,r)=\int d\rho_l^{(1)}(E')\varphi_l(k',r)\times\int_0^r dr'\varphi_l^{(1)}(k',r')\varphi_l^{(1)}(k,r'), \tag{8.1}$$

where the quantities with the superscript "1" refer to a given potential $V^{(1)}(r)$, and those without superscript, to another potential $V(r)$. If we insert (4.19) and (7.10) in (8.1) and use (4.1), we obtain

$$I=\frac{i}{\pi}\int_{-\infty}^\infty\frac{dk'k'^{l+1}}{k^2-k'^2}\frac{\varphi_l(k',r)}{f_l^{(1)}(k')}W[\varphi_l^{(1)}(k,r),f_l^{(1)}(k',r)]$$
$$+\sum_n\frac{\varphi_l(-i\kappa_n, r)}{k^2+\kappa_n^2}\frac{1}{N_n^2}W[\varphi_l^{(1)}(k,r),\varphi_l^{(1)(n)}(r)], \tag{8.2}$$

if κ_n refers to the bound states of $V^{(1)}(r)$ and we take k slightly off the real axis into the lower half-plane. Adding to the integral a similar one over a large semicircle in the lower half-plane, we can evaluate it by means of Cauchy's residue theorem. The result exactly cancels the bound state sum in (8.2). Thus we are left with the negative of the integral over the large semicircle where we may use the asymptotic functions for large k', given by (3.13), (3.17), and (4.16). The result is that

$$I(E,r)=\varphi_l(k,r)-\tfrac{1}{2}\varphi_l^{(1)}(k,r). \tag{8.3}$$

The next step is to notice that the completeness proof, i.e., the derivation of (7.8), would have gone through just as well if $\varphi_l(k,r')$ had been replaced by $\varphi_l^{(1)}(k,r')$. If in the resulting formula we set $h(r)=\varphi_l^{(1)}(k,r)$, we get

$$\tfrac{1}{2}\varphi_l^{(1)}(k,r)=\int d\rho_l(E')\varphi_l(k',r)\times\int_0^r dr'\varphi_l^{(1)}(k,r')\varphi_l^{(1)}(k',r'). \tag{8.4}$$

The implication of (8.1), (8.3), and (8.4) is that

$$\varphi_l(k,r)=\varphi_l^{(1)}(k,r)+\int_0^r dr'K_l(r,r')\varphi_l^{(1)}(k,r'), \tag{8.5}$$

where

$$K_l(r,r')=\int d[\rho_l^{(1)}(E)-\rho_l(E)]\varphi_l(k,r)\varphi_l^{(1)}(k,r'). \tag{8.6}$$

Equation (8.5) resembles that containing a complete Green's function; however, in contrast to the latter, $K_l(r,r')$ has the remarkable property of being independent of the energy. It obviously satisfies the differential equation

$$\frac{\partial^2}{\partial r^2}K_l(r,r')-\left[V(r)+\frac{l(l+1)}{r^2}\right]K_l(r,r')$$
$$=\frac{\partial^2}{\partial r'^2}K_l(r,r')-\left[V^{(1)}(r')+\frac{l(l+1)}{r'^2}\right]K_l(r,r'). \tag{8.7}$$

Inserting (8.5) in the Schrödinger equation and using (8.7) readily leads to

$$2\frac{d}{dr}K_l(r,r)=V(r)-V^{(1)}(r). \tag{8.8}$$

In addition, K_l satisfies the boundary condition

$$K_l(0,r)=0. \tag{8.9}$$

If we multiply, finally, (8.5) by $\varphi_l^{(1)}(k,r'')$ and integrate with the weight $\rho_l^{(1)}-\rho_l$, we obtain the Gel'fand Levitan integral equation

$$K_l(r,r')=g_l(r,r')+\int_0^r dr''K_l(r,r'')g_l(r'',r'), \tag{8.10}$$

where

$$g_l(r'',r')=\int d[\rho_l^{(1)}(E)-\rho_l(E)]\times\varphi_l^{(1)}(k,r'')\varphi_l^{(1)}(k,r'). \tag{8.11}$$

[51] I. M. Gel'fand and B. M. Levitan, Doklady Akad. Nauk S.S.S.R. **77**, 557 (1951).
[52] I. M. Gel'fand and B. M. Levitan, Izvest. Akad. Nauk S.S.S.R. **15**, 309 (1951).
[53] The procedure follows Jost and Kohn.[54] See also N. Levinson, Phys. Rev. **89**, 755 (1953).
[54] R. Jost and W. Kohn, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **27**, No. 9 (1953).

It can be shown[54] that (8.10) always has a unique solution. A knowledge of the spectral function $\rho_l(E)$ thus determines $K_l(r,r')$ via (8.10) and (8.11); the function $\varphi_l(k,r)$ then follows from (8.5), and the potential, from (8.8). The spectral function in turn is given by (7.10) in terms of the bound state energies and the Jost function $f_l(k)$, and the latter is given by (5.21) in terms of the phase shift and the bound state energies. This demonstrates that in general, binding energies and scattering phaseshifts are completely independent, and that if there are n_l bound states, then there exists an n_l parameter (the N_n) family of potentials all of which lead to the same phaseshift and to the same binding energies.

A simple application is that in which the difference $\rho_l^{(1)}-\rho_l$ is infinitesimal, due to an infinitesimal change in the phaseshift.[55] In that instance one uses (5.21) and (7.11) in (8.6) and finds

$$K_l(r,r')=-\frac{2}{\pi}\int_{-\infty}^{\infty}dk'k'\delta\delta_l(k')\times\left[\mathfrak{G}_l^{(P)}(E';r,r')-\sum_n\frac{\psi_l^{(n)}(r)\psi_l^{(n)}(r')}{E'-E_n}\right]$$

to first order in the variation $\delta\delta_l(k)$ of the phaseshift. Consequently, by (8.8),

$$\frac{\delta V(r)}{\delta\delta_l(k)}=-\frac{4}{\pi}k\frac{d}{dr}\left\{\mathfrak{G}_l^{(P)}(E;r,r)-\sum_n\frac{[\psi_l^{(n)}(r)]^2}{E-E_n}\right\}. \quad (8.12)$$

Specifically, for $r=0$, we have by (6.3), (3.3), and (4.3')

$$\frac{d}{dr}\mathfrak{G}_l^{(P)}(E;r,r)\underset{r\to 0}{\longrightarrow}-\frac{1}{2l+1},$$

and therefore

$$\delta V(0)/\delta\delta_l(k)=4k/\pi(2l+1). \quad (8.13)$$

This equation can be integrated immediately

$$V(0)-V^{(1)}(0)=[8/\pi(2l+1)]\int_0^{\infty}dkk[\delta_l(k)-\delta_l^{(1)}(k)],$$

where $V(r)$ and $V^{(1)}$ must have the same bound states. Finally we use the Bargmann potentials (Sec. 10) in order to construct a potential $V^{(1)}$ with the same bound states as V and whose phaseshift is asymptotically equal to $\delta_l(k)$, i.e., the value given by (5.1'). The result is a simple exact relation between the value of the potential at the origin and the lth phaseshift and bound state energies $E_n^{(l)}(<0)$[55]:

$$V(0)=\frac{4}{2l+1}\left\{\frac{2}{\pi}\int_0^{\infty}dk\left[k\delta_l(k)+\frac{1}{2}\int_0^{\infty}drV(r)\right]-\sum_n E_n^{(l)}\right\}, \quad (8.14)$$

or

$$V(0)=\frac{4}{2l+1}\left\{-\frac{2}{\pi}\int_0^{\infty}dkk[\delta_l(k)+k\delta_l'(k)]-\sum_n E_n^{(l)}\right\}, \quad (8.14')$$

the prime indicating differentiation with respect to k. This shows that although the phaseshifts of the same potential are asymptotically equal for different l values and near $k=0$ become smaller as l increases, their first moments *increase* with growing l.

9. GENERALIZATION TO THE CASE WITH COUPLING

Almost everything done in the preceding sections can be generalized to the case in which the potential $V_{ls,l's'}{}^J$ in (2.9) has off-diagonal elements.[56] It is then most convenient to write (2.10) in matrix notation suppressing the indices; thus

$$-(d^2/dr^2)\Psi_J+V^J\Psi_J+[L(L+1)/r^2]\Psi_J=k^2\Psi_J, \quad (9.1)$$

where V^J is the square matrix (2.9), L is the diagonal matrix of the l values, and Ψ_J is the square matrix $\Psi_{ls,l's'}{}^J$ of (2.6). It may be well to recall the meaning of this square matrix: Each column is a solution of (9.1), its components indicating the various angular momentum components; the columns differ from one another by their boundary conditions, e.g., by the incoming wave according to (2.8). It is more convenient to work with such a square matrix then with the individual columns.

The fact that (9.1) has equations of different angular momenta coupled together leads to certain complications owing to the different behavior at $r=0$ of the solutions belonging to different l values. We want to introduce a regular solution $\Phi_J(k,r)$ which would be the generalization of $\varphi_l(k,r)$. However, the boundary condition (3.3) cannot be generalized in any simple way.[58] It is easier to write down directly the matrix integral equation that is to replace (3.7). But unless special precautions are taken or else a very strong assumption is made concerning the behavior of the off-diagonal elements of V^J, the resulting integral diverges at $r=0$. This divergence can be eliminated by adding a judicious inhomogeneity in the integral equation. We shall restrict ourselves to the case of $s=s'=1$ with tensor force coupling. The procedure is readily generalized to higher spin values.

[55] R. G. Newton, Phys. Rev. **101**, 1588 (1956).

[56] The content of this section follows Newton and Jost,[57] and Newton.[17] The order of the matrices, however, has been changed. Equations in footnote references 17 and 57 have to be read from right to left in order to agree with those in this section.

[57] R. G. Newton and R. Jost, Nuovo cimento **1**, 590 (1955).

[58] A simple example of a square well V^J furnishes an illustration; see W. Rarita and J. Schwinger, Phys. Rev. **59**, 436 (1941). If one wants to solve the equations by series expansion, even the regular solution contains the logarithmic terms of the Fuchs theory; cf., e.g., E. L. Ince, *Ordinary Differential Equations* (Longmans, Green and Co., Ltd., New York, 1927), p. 356 ff.

If we write

$$U_J(k,r)\equiv\begin{pmatrix} k^{-J}u_{J-1}(kr) & 0 \\ 0 & k^{-J-2}u_{J+1}(kr)\end{pmatrix}, \quad (9.2)$$

$$\mathcal{G}_J(k;r,r')\equiv\begin{pmatrix} g_{J-1}(k;r,r') & 0 \\ 0 & g_{J+1}(k;r,r')\end{pmatrix}, \quad (9.3)$$

with $g_l(k;r,r')$ given by (3.6), then we can define a regular matrix solution $\Phi_J(k;r)$ of (9.1) by the integral equation

$$\Phi_J(k,r)=U_J(k,r)\left[1+(2J+1)\int_1^r dr'r'^{-1}V_T{}^J(r')\right]$$
$$+\int_0^r dr'[\mathcal{G}_J(k;r,r')V^J(r')\Phi_J(k,r')$$
$$-(2J+1)U_J(k,r)V_T{}^J(r')], \quad (9.4)$$

where

$$V_T{}^J\equiv\begin{pmatrix} 0 & 0 \\ V_{J-1,J+1}{}^J & 0\end{pmatrix}.$$

This integral equation can always be solved by successive approximations if the elements of V^J satisfy (3.1a); the matrix function $\Phi_J(k,r)$ has all the regularity and reality properties of $\varphi_l(k,r)$.

The generalization of the solution $f_l(k,r)$ is a matrix function $F_J(k,r)$ defined by the boundary condition

$$\lim_{r\to\infty} e^{ikr}F_J(k,r)=i^L, \quad (9.5)$$

or the integral equation

$$F_J(k,r)=W_J(kr)$$
$$-\int_r^\infty dr'\,\mathcal{G}_J(k;r,r')V^J(r')F_J(k,r') \quad (9.6)$$

with

$$W_J(kr)=\begin{pmatrix} w_{J-1}(kr) & 0 \\ 0 & w_{J+1}(kr)\end{pmatrix}.$$

Under the hypothesis (3.1b) on all elements of the potential matrix this integral equation can also always be solved by successive approximations. $F_J(k,r)$ has all the regularity properties of $f_l(k,r)$.

The generalized Jost function $F_J(k)$ is defined by the analog of (4.3)

$$F_J(k)\equiv k^LW[F_J(k,r),\Phi_J(k,r)], \quad (9.7)$$

where the Wronskian matrix[59]

$$W[F,\Phi]\equiv F^T\Phi'-F^{T\prime}\Phi$$

is defined so that it is independent of r if F and Φ both solve the same Eq. (9.1). In terms of $F_J(k)$ we have

$$\Phi_J(k,r)=\tfrac{1}{2}i[F_J(k,r)F_J{}^T(-k)$$
$$-(-)^LF_J(-k,r)F_J{}^T(k)]k^{-L-1} \quad (9.8)$$

instead of (4.1). The matrix function $F_J(k)$ has all the regularity properties of $f_l(k)$.

Comparison of the asymptotic form of (9.8) by (9.5) with (2.23) then gives us the S matrix

$$S^J(k)=F_J{}^T(k)[F_J{}^T(-k)]^{-1}. \quad (9.9)$$

This can be transformed by using the fact that because of the boundary condition

$$W[\Phi_J(k,r),\Phi_J(k,r)]=0.$$

If (9.8) is inserted in this one obtains

$$F_J(-k)F_J{}^T(k)=F_J(k)F_J{}^T(-k), \quad (9.10)$$

which shows that (9.9) can also be written

$$S^J(k)=[F_J(k)]^{-1}F_J(k), \quad (9.9')$$

at the same time verifying the symmetry of S^J. Since $F_J(k)$ has the property (4.7) it follows also that S^J is unitary. Furthermore, (9.9′) implies that

$$S^J(-k)=[S^J(k)]^{-1}. \quad (9.11)$$

The relation of the physical wave function Ψ_J to Φ_J is seen by comparing (9.8) with (2.23), together with (9.5) and (9.9):

$$\Psi_J(k,r)=\Phi_J(k,r)k^{L+1}[F_J{}^T(-k)]^{-1}. \quad (9.12)$$

This is the analog of (4.8).

An integral representation for $F_J(k)$ can again be written down, but it is complicated by the extra inhomogeneities in (9.4). $F_J(k)$ being a matrix, it is not related in any direct way to the Fredholm determinant of (2.8).

The bound states can again be obtained from $F_J(k)$. This time they are those points $k=k_0$ in the lower half plane where $\det F_J(k_0)=0$. Why that is so is most easily understood by introducing an auxiliary irregular solution $I_J(k,r)$, which satisfies

$$W[I_J,\Phi_J]=1,\quad W[I_J,I_J]=0, \quad (9.13)$$

and which, for all fixed $r\neq 0$, is an entire function of k^2.[17] We can then express $F_J(k,r)$ in terms of $\Phi_J(k,r)$ and $I_J(k,r)$,

$$F_J(k,r)=\Phi_J(k,r)k^LF_J{}'(k)+I_J(k,r)k^{-L}F_J(k), \quad (9.14)$$

where

$$F_J{}'(k)\equiv -k^{-L}W[F_J,I_J].$$

In contrast to (9.8), (9.14) always holds in the lower half of the complex k plane, too.

Now, if $\det F_J(k_0)=0$ and $\mathrm{Im}k_0<0$, then there exists a constant vector a so that $F_J(k_0)a=0$. Equation (9.14)

[59] A superscript "T" indicates the transposed matrix. The symmetry of the potential matrix is an important assumption. By (2.17) it follows from time reversal invariance of the interaction H_I.

then shows that

$$F_J(k_0,r)a=\Phi_J(k_0,r)k_0{}^L F_J'(k_0)a \qquad (9.15)$$

is a solution regular at the origin and exponentially decreasing at infinity; k_0^2 is thus a discrete eigenvalue and $F_J(k_0,r)a$ is the corresponding eigenfunction. Again it follows that k_0 must lie on the negative imaginary axis. Conversely, if k_0^2 is a discrete eigenvalue, then there must exist a constant vector a so that (9.15) holds and hence by (9.14)

$$I_J(k_0,r)k_0{}^{-L}F_J(k_0)a=0$$

identically in r. It then follows from (9.13) that $F_J(k_0)a=0$ and consequently $\det F_J(k_0)=0$.

The significance of the vector a is shown by (9.15). By the boundary condition (9.5) the asymptotic form of the bound-state wave function is proportional to

$$\exp(-|k_0|r)(a_1,-a_2).$$

In that sense the ratio of the components of a determines the mixture of angular momenta that forms a bound state. It is always possible that accidentally more than one mixture is bound with the same energy; that is the degenerate case. In the present instance of only two coupled angular momenta it would imply that $F_J(k_0)=0$.

It can be proved[17,57] that if $\det F_J(k_0)=0$ when $\operatorname{Im}k_0<0$, then $[F_J(k)]^{-1}$ has exactly a simple pole at $k=k_0$. That statement has no bearing on the question of degeneracy. The contrary is true for $\det F_J(k)$, which in the degenerate case has a double zero and hence its inverse, a double pole.

The point $k=0$ is somewhat complicated. For $J>1$, $\det F_J(0)=0$ implies a zero-energy bound state; for $J=1$ it does so only if $[F_J(0)]_{22}=0$ and $k^2[F_J(k)]_{12}\to 0$ as $k\to 0$. The matrix function $k^{-L}F_J(k)k^L$ is continuous at $k=0$, and the analog of (4.25) is that

$$Q_J\equiv\lim_{k\to 0}k^{2-L}[F_J(k)]^{-1}k^L \qquad (9.16)$$

always exists and differs from zero if and only if $E=0$ is a discrete eigenvalue.

The argument concerning the finiteness of the number of zeros of $f_l(k)$ in the lower half-plane can be carried over directly to $F_J(k)$. Again the result is that the number of bound states for a given J is finite if all elements of V^J satisfy (3.1).

At high energies we have the analog of (4.16). For $\operatorname{Im}k\leq 0$

$$\lim_{|k|\to\infty}F_J(k)=1 \qquad (9.17)$$

and consequently,

$$\lim_{k\to\pm\infty}S^J(k)=1.$$

The statements made in Secs. 4 and 5 concerning the high-energy behavior in the upper half of the complex plane under stronger assumptions on the potential carry over to the present case.

At low energies one can generalize first of all (5.15). If we define

$$2i\eta_J(k)\equiv\log\det S^J(k) \qquad (9.18)$$

then comparison with (2.20) shows that η_J is the sum of the eigenphaseshifts of total angular momentum J

$$\eta_J(k)=\textstyle\sum_\alpha \delta_\alpha{}^J(k). \qquad (9.18')$$

One can then show that[57]

$$\eta_J(0)-\eta_J(\infty)=\begin{cases}\pi(n_J+\frac{1}{2}), & \text{if } J=1 \text{ and } k=0 \text{ is a resonance,}\\ \pi n_J, & \text{otherwise,}\end{cases} \qquad (9.19)$$

n_J being the number of bound states of total angular momentum J (counted twice in the degenerate case); the resonant case is that in which $\det F_1(0)=0$ and $Q_1=0$ [see (9.16)].

The way in which S^J approaches its zero-energy value is found similarly as in the case of no coupling. The result is that, provided the $(2J+4)$-th absolute moments of all elements of V^J exist and they are absolutely integrable, the generalization of (5.19) is

$$S^J(k)-1=\begin{pmatrix}0(k^{2J-1}) & 0(k^{2J+1})\\ 0(k^{2J+1}) & 0(k^{2J+3})\end{pmatrix} \text{ as } k\to 0 \qquad (9.20)$$

unless $\det F_J(0)=0$; in the latter case we have

$$S^J(k)-1=\begin{pmatrix}0(k^{2J-3}) & 0(k^{2J-1})\\ 0(k^{2J-1}) & 0(k^{2J+1})\end{pmatrix}, \text{ if } J>1,$$
$$=\begin{pmatrix}0(k) & 0(k^3)\\ 0(k^3) & 0(k^3)\end{pmatrix}, \text{ if } J=1, \qquad (9.20')$$

unless we have the "resonance case."

We may now write down the complete Green's function which solves

$$\left[-\frac{d^2}{dr^2}+V^J(r)+\frac{L(L+1)}{r^2}-k^2\right]\mathfrak{G}_J(k;r,r')=-\delta(r-r'). \qquad (9.21)$$

The arguments leading to its construction are the same as in Sec. 6. The result is the analog of (6.3),

$$\mathfrak{G}_J(k;r,r')=\begin{cases}(-)^J\Phi_J(k,r)k^L[F_J{}^T(-k)]^{-1}F_J{}^T(-k,r'), & r<r',\\ (-)^J F_J(-k,r)[F_J(-k)]^{-1}k^L\Phi_J{}^T(k,r'), & r>r',\end{cases} \qquad (9.22)$$

or by (9.12)

$$\mathfrak{G}_J(k;r,r')=\begin{cases}(-)^J\Psi_J(k,r)F_J{}^T(-k,r'), & r<r'\\ (-)^J F_J(-k,r)\Psi_J{}^T(k,r'), & r>r'.\end{cases} \qquad (9.22')$$

The verification that this is indeed a Green's function, i.e., that it is continuous at $r=r'$ and fulfills the matrix version of (6.2) is not completely trivial. It rests on the observation that[60]

$$F_J(k,r)F_J^T(-k,r)-F_J(-k,r)F_J^T(k,r)=0,$$
$$F_J'(k,r)F_J^T(-k,r)-F_J'(-k,r)F_J^T(k,r)=(-)^J 2ik. \quad (9.23)$$

These equations are proved by introducing an auxiliary matrix solution $\Lambda(k,r)$ of (2.10), which satisfies the boundary condition

$$\Lambda(k,r_0)=0,\quad \Lambda'(k,r_0)=1,$$

at an arbitrary point $r_0\neq 0$. Λ can be expressed in terms of $F_J(k,r)$ and $F_J(-k,r)$, the coefficients being found by evaluating the Wronskians. If we then insert the boundary condition at r_0 we obtain (9.23).

The regularity properties of $\mathfrak{G}_J$ are the same as those of $\mathfrak{G}_l$. We can again relate the S matrix to it by solving (2.8):

$$\Psi_J(k,r)=k^{L+1}U_J(k,r)+\int_0^\infty dr'\mathfrak{G}_J(k;r,r')V^J(r')U_J(k,r')k^{L+1} \quad (9.24)$$

and then inserting this solution in (2.21)

$$S^J(k)=1-2ik^{L+\frac{1}{2}}\int_0^\infty drU_J(k,r)V^J(r)U_J(k,r)k^{L+\frac{1}{2}}-2ik^{L+\frac{1}{2}}\int_0^\infty dr\int_0^\infty dr'U_J(k,r)V^J(r)\times\mathfrak{G}_J(k;r,r')V^J(r')U_J(k,r')k^{L+\frac{1}{2}}. \quad (9.25)$$

The completeness of the eigenfunction of (9.1) is proved[17,57] by the same method as in Sec. 7 for a single equation. The result is that (7.9′) is replaced by

$$\int\Phi_J(E,r)d\mathrm{P}_J(E)\Phi_J^T(E,r')=\delta(r-r'), \quad (9.26)$$

where the spectral function is given by

$$\frac{d\mathrm{P}_J(E)}{dE}=\begin{cases}\frac{2\mu}{\pi\hbar^2}k^{L+\frac{1}{2}}[F_J(k)F_J^T(-k)]^{-1}k^{L+\frac{1}{2}}, & E>0,\\ \sum_n C_n\delta(E-E_n), & E\leq 0,\end{cases} \quad (9.27)$$

with $\mathrm{P}_J(-\infty)=0$. $\mathrm{P}_J(E)$ is a real, symmetric, positive semidefinite matrix function of E. The matrices C_n are real symmetric, positive semidefinite, and in general singular, with the property

$$C_n=\int_0^\infty drC_n\Phi_J^{(n)T}(r)\Phi_J^{(n)}(r)C_n, \quad (9.28)$$

where

$$\Phi_J^{(n)}(r)\equiv\Phi_J(-i\kappa_n,r).$$

We can always write

$$C_n=\alpha_n^2B_n,$$

where α_n is a real number and B_n, a real symmetric projection[61]:

$$B_n=b_n\times b_n$$

in terms of the "vector"

$$b_n=(1+\beta_n^2)^{-\frac{1}{2}}(1,\beta_n).$$

We also define a vector

$$\psi_J^{(n)}(r)\equiv\alpha_n(1+\beta_n^2)^{-\frac{1}{2}}\times[\Phi_{J11}^{(n)}(r)+\beta_n\Phi_{J12}^{(n)}(r),\Phi_{J21}^{(n)}(r)+\beta_n\Phi_{J22}^{(n)}(r)]$$

with the property

$$\int_0^\infty dr|\psi_J^{(n)}(r)|^2=1,$$

which follows from (9.28). With these definitions we have

$$\Phi_J^{(n)}(r)C_n\Phi_J^{(n)T}(r')=\psi_J^{(n)}(r)\times\psi_J^{(n)}(r').$$

The completeness (9.26) thus can also be written in terms of the physical wave function and normalized bound-state wave functions:

$$\frac{2}{\pi}\int_0^\infty dk\Psi_J(k,r)\Psi_J^\dagger(k,r')+\sum_n\psi_J^{(n)}(r)\times\psi_J^{(n)}(r')=\delta(r-r'). \quad (9.26')$$

Similarly, for the complete Green's function

$$\mathfrak{G}_J(E;r,r')=\frac{\hbar^2}{2\mu}\int\frac{\Phi_J(E',r)d\mathrm{P}_J(E')\Phi_J^T(E',r)}{E-E'}, \quad (9.29)$$

or

$$\mathfrak{G}_J(k;r,r')=\frac{1}{\pi}\int_{-\infty}^\infty\frac{dk'}{k'}\frac{\Psi_J(k',r)\Psi_J^\dagger(k',r')}{k-k'}+\sum\frac{\psi_J^{(n)}(r)\times\psi_J^{(n)}(r')}{k^2+\kappa_n^2}. \quad (9.29')$$

The Gel'fand Levitan equations can be generalized to the case with coupling in a straight forward manner.[57] The result is that (8.5)–(8.7), and (8.11) are replaced by

$$\Phi_J(k,r)=\Phi_J^{(1)}(k,r)+\int_0^\infty dr'K_J(r,r')\Phi_J^{(1)}(k,r'), \quad (9.30)$$

$$K_J(r,r')=\int\Phi_J(k,r)\times d[\mathrm{P}_J^{(1)}(E)-\mathrm{P}_J(E)]\Phi_J^{(1)T}(k,r'), \quad (9.31)$$

[60] Notice the position of the transposed functions. These are *not* Wronskians and their constancy is not a simple consequence of the differential equation.

[61] The cross denotes a direct product: $B_{ij}=b_ib_j$.

$$\frac{\partial^2}{\partial r^2}K_J(r,r')-\left[V^J(r)+\frac{L(L+1)}{r^2}\right]K_J(r,r')$$
$$=\frac{\partial^2}{\partial r'^2}K_J(r,r')-K_J(r,r')\left[V^{(1)J}(r')+\frac{L(L+1)}{r'^2}\right], \quad (9.32)$$

$$g_J(r,r')=\int\Phi_J^{(1)}(k,r)$$
$$\times d[\mathrm{P}_J^{(1)}(E)-\mathrm{P}_J(E)]\Phi_J^{(1)T}(k,r'), \quad (9.33)$$

while (8.8) to (8.10) retain the same form as before.

The potential matrix V^J is thus determined from the spectral function $\mathrm{P}_J(E)$ in the same manner as in the case of no coupling. However, it is now no longer so simple to infer the generalized Jost function $F_J(k)$ and thus $\mathrm{P}_J(E)$ from $S^J(k)$ and the bound states. The procedure leading to (5.21) cannot be generalized, the logarithm of a matrix not being well defined.[62] The problem was solved by Newton and Jost[57] with the result that, in contrast to the case of no coupling, not all matrix functions $S^J(k)$ admit of a splitup (9.9′) with $F_J(k')$ having all the required properties. No simple way is known to determine whether or not a given $S^J(k)$ leads to an $F_J(k)$, except to solve the integral equation of footnote reference 57 in order to find F_J.

10. EXAMPLES

(a) Square Well

In the simple case of a square well[63] one readily finds that the Jost function is

$$f_l(k)=(k/K)^l[w_l(kr_0)u_l'(Kr_0)$$
$$-(k/K)u_l(Kr_0)w_l'(kr_0)], \quad (10.1)$$

where r_0 is the radius of the potential of strength V_0, $K^2=k^2-V_0$, and the prime indicates differentiation with respect to the argument of the function. In the case $l=0$ we have

$$f_0(k)=(r_0K)^{-1}e^{-ikr_0}\sin(Kr_0)g(-ikr_0),$$

where, with

$$z=-ikr_0,\quad z_0^2=-r_0^2V_0,\quad \zeta^2=z_0^2-z^2\equiv(\xi^2+i\eta)^2,$$

we write

$$g(z)=\zeta\cot\zeta-z.$$

The zeros of $f_0(k)$ are found from those of $g(z)$. The real roots with $z<z_0$ are determined by the intersection of the two curves

$$z=\xi\cot\xi,\quad z=(z_0^2-\xi^2)^{\frac{1}{2}},$$

shown in Fig. 1 with some intersections for $z_0^2>0$, i.e., an attractive potential. An intersection for negative z means a negative imaginary root of $f_0(k)$ and hence, a bound state. Such a root evidently exists whenever $z_0>\frac{1}{2}\pi$. When $1<z_0<\frac{1}{2}\pi$, then there is an intersection for positive z, i.e., a positive imaginary root of $f_0(k)$. We then have a "virtual bound state." When $z_0<1$ then we must replace ξ by $i\eta$ and the curves become

$$z=\eta\coth\eta,\quad z=(z_0^2+\eta^2)^{\frac{1}{2}}.$$

The intersection keeps moving up and there is always a virtual bound state.

For $z_0^2<0$ (i.e., a repulsive potential), there is obviously no intersection and we never have a virtual bound state.

The complex zeros of $f_0(k)$ are obtained from those roots of

$$H(z)\equiv g(z)g(-z)=\csc^2\zeta(\zeta-z_0\sin\zeta)(\zeta+z_0\sin\zeta)\equiv h(\zeta),$$

which lie in the right half of the complex z plane. A zero ζ_0 of $h(\zeta)$ must satisfy the equations

$$\xi_0\cot\xi_0=\eta_0\coth\eta_0$$
$$\xi_0=\pm z_0\sin\xi_0\cosh\eta_0$$

if the potential is attractive, or

$$\xi_0\tan\xi_0=-\eta_0\tanh\eta_0$$
$$\eta_0=\pm z_0\sin\xi_0\cosh\eta_0$$

if it is repulsive. They can be shown to have infinitely many solutions.[64]

(b) Zero-Range Potential

The case of a potential of zero range is included here only for the sake of cautioning the unwary. If the potential vanishes identically for $r>R$ then the wave function in the outside region is determined by assigning it at $r=R$ a given logarithmic derivative c which becomes less and less energy dependent the shorter the potential range R. In the limit as $R\to0$, then, the

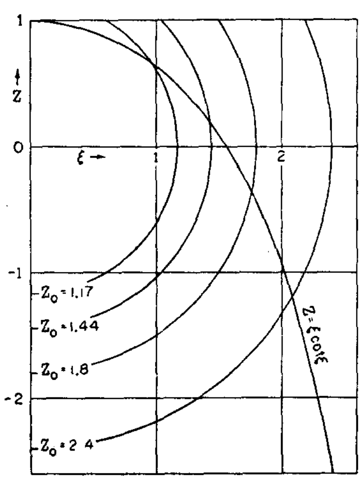

FIG. 1. Bound and virtual bound states in a square well; $z=-ikr_0$, $z_0^2=-r_0^2V_0$. The z coordinate of the intersection of the curve $z=\xi\cot\xi$ with the circle $z^2+\xi^2=z_0^2$ gives the energy of the bound state, if negative, or of the virtual bound state, if positive.

[62] It is not known whether $F_J(k)$ is diagonalizable or, if it is, whether its eigenvalues and diagonalizing matrix separately are analytic functions.

[63] This case was treated in great detail by Nussenzweig.[64] The procedure below is similar to his.

[64] H. M. Nussenzweig, Nuclear Phys. 3, 499 (1959).

potential is replaced by the boundary condition[65]

$$\lim_{r\to 0}\varphi'(k,r)/\varphi(k,r)=c,$$

while (3.3) is discarded. The function $f(k,r)$ is in this case simply the free function e^{-ikr} and from (4.3) [Eq. (4.3′) no longer holds],

$$f(k)=c+ik.$$

This function vanishes at $k=ic$ so that if c is negative, then there is a bound state of energy $-\hbar^2c^2/2\mu$, and if c is positive, then there is a virtual bound state. Notice that (4.16) is now no longer true.

The S matrix is

$$S(k)=(c+ik)/(c-ik), \tag{10.2}$$

so that $S(\infty)=-1$. The Levinson theorem (5.15) is also violated since now

$$\delta(0)-\delta(\infty)=\pm\tfrac{1}{2}\pi. \tag{10.3}$$

depending on the sign of c. The explanation[66] of this fact is that the limit of zero range is not uniform in k, as can be seen explicitly by writing

$$\tan\delta=(k\cot kR-c)/(k+c\cot kR).$$

If we let $k\to\infty$ then we know that for fixed R, c approaches its free value $k\cot kR$, and hence $\tan\delta\to 0$; we may subsequently let $R\to 0$ and get no change. But if we let $R\to 0$ with c fixed, we get

$$\tan\delta=k/c$$

and c is independent of k. If we now let $k\to\infty$ we get the anomalous value $\tan\delta=\infty$.

(c) Repulsive Core

If the potential is positive infinite for $r<R_c$, then the boundary condition (3.3) is replaced by

$$\varphi_l(k,R_c)=0,$$
$$\varphi_l'(k,R_c)=1,$$

for each l. The solution $f_l(k,r)$ is completely unaware of the core for $r>R_c$. The analyticity properties are thus quite unchanged. The Jost function is given by (4.3), but instead of (4.3′), we get

$$f_l(k)=k^l f_l(k,R_c),$$

and hence by (3.17), as $|k|\to\infty$ in the lower half-plane or on the real axis,

$$f_l(k)=(ik)^l e^{-ikR_c}+o(k^l e^{\nu R_c}).$$

The S matrix is

$$S_l(k)=(-)^l[f_l(k,R_c)/f_l(-k,R_c)].$$

Thus as $|k|\to\infty$ for real k

$$\delta_l(k)\sim -kR_c+\tfrac{1}{2}\pi l+o(1).$$

In other words, the phaseshift no longer tends to a multiple of 2π at high energies, but instead keeps increasing linearly.

The S matrix elements of the first three angular momenta for a *pure* repulsive core are

$$\begin{aligned}S_0(k)&=\exp(-2ikR_c),\\ S_1(k)&=-\exp(-2ikR_c)\cdot\frac{k-iR_c^{-1}}{k+iR_c^{-1}},\\ S_2(k)&=\exp(-2ikR_c)\cdot\frac{(kR_c)^2-3ikR_c-3}{(kR_c)^2+3ikR_c-3}.\end{aligned} \tag{10.4}$$

Whereas for $l=0$, S_l is an entire function, for $l\geq 1$ it has poles in the lower half-plane, i.e., on the second sheet of the Riemann surface as a function of the energy.

Although the Levinson theorem (5.15) is not true when a repulsive core is present, one can prove a similar theorem for the difference between the actual phaseshift and the pure core phaseshift for the same core radius.

(d) Exponential Potential

If the potential has the form

$$V(r)=-V_0e^{-r/a},$$

then the s wave radial equation is explicitly solvable by setting $x=e^{-r/a}$. The result is that[12,67,68]

$$f_0(k,r)=\exp[-iak\log(a^2V_0)]\times\Gamma(1+2iak)J_{2iak}(2aV_0^{\frac{1}{2}}e^{-r/2a}) \tag{10.5}$$

and the Jost function

$$f_0(k)=\exp[-iak\log(a^2V_0)]\times\Gamma(1+2iak)J_{2iak}(2aV_0^{\frac{1}{2}}). \tag{10.6}$$

The points where

$$J_{2iak}(2aV_0^{\frac{1}{2}})=0$$

determine the bound states (for $\mathrm{Im}k_0<0$) and the virtual states as well as the "resonances."[69]

The function $f_0(k)$ has infinitely many simple poles on the positive imaginary axis because of the gamma function. They occur at $k=in/2a$ for all positive integers n. There is the exceptional possibility that $E=-\hbar^2n^2/8a^2\mu$ is the energy of a bound state. Since $J_{-n}(z)=(-)^nJ_n(z)$ the Bessel function then vanishes at the same point where the gamma function has a pole. In that case $f_0(k)$ does not have a pole at $k=in/2a$;

[65] This works only for $l=0$.

[66] This remark is due to R. E. Peierls; private communication.

[67] H. A. Bethe and R. Bacher, Revs. Modern Phys. **8**, 111 (1936).

[68] S. T. Ma, Phys. Rev. **69**, 668 (1946).

[69] It follows incidentally that $J_\nu(z)$ can have no real zeros for $\mathrm{Re}\nu>0$ unless $\mathrm{Im}\nu=0$. This does not appear to be a known property of Bessel functions.

but since it still has a zero at $k=-in/2a$, the S matrix still retains its pole.

It may be expected from the structure of $f_0(k,r)$ for the pure exponential potential that the appearance of poles in $f_0(k)$ at $k=in/2a$ for positive integers n is a general feature of potentials whose asymptotic tail is proportional to $e^{-r/a}$. This has indeed been demonstrated recently by Peierls.[70]

(e) Yukawa-Type Potentials

Suppose that the potential can be written in the form

$$V(r)=V_0\int d\alpha\rho(\alpha)e^{-\alpha r}, \tag{10.7}$$

where $\rho(\alpha)=0$ for $\alpha<\mu$, $\mu>0$.[71] A special case is the Yukawa potential, obtained by setting $\rho(\alpha)=\text{const.}$ for $\alpha>\mu$. We shall examine the implications of the foregoing form of V for $l=0$ only.[12,15,26,70,72]

If we write

$$g(k,r)\equiv f_0(k,r)e^{ikr},$$

then the integral equation (3.8) becomes for the potential (10.7)

$$g(k,r)=1+(V_0/2ik)\int d\alpha\rho(\alpha)\int_0^\infty dr' \times(1-e^{-2ikr'})e^{-\alpha(r+r')}g(k,r+r'),$$

and the Jost function is

$$f_0(k)=g(k,0).$$

We solve the integral equation by iteration

$$g(k,r)=\sum_{n=0}^{\infty}g_n(k,r),$$

and easily find that

$$g_0(k,r)=1,$$

$$g_n(k,r)=V_0{}^n\int\frac{d\alpha_1}{\alpha_1}\frac{\rho(\alpha_1)}{\alpha_1+2ik}\int\frac{d\alpha_2}{\alpha_2}\frac{\rho(\alpha_2-\alpha_1)}{\alpha_2+2ik}\cdots \times\int\frac{d\alpha_n}{\alpha_n}\frac{\rho(\alpha_n-\alpha_{n-1})}{\alpha_n+2ik}e^{-\alpha_n r},\quad n\geq 1. \tag{10.8}$$

It is clear from this that in general $f_0(k,r)$ will have a branch cut along the positive imaginary axis starting at $k=\frac{1}{2}i\mu$ and running to infinity. Moreover, if we assume that $\rho(\alpha)$ is bounded

$$|\rho(\alpha)|\leq M,$$

then $f_0(k,r)$ is a regular analytic function in the entire k plane, except for the cut; at $k=\frac{1}{2}i\mu$ it has a logarithmic singularity (unless $\rho(\mu)=0$), while everywhere else on the cut it is continuous. It is readily seen from the foregoing that if k remains a finite distance away from the cut, i.e., if

$$|\mathrm{Re}k|\geq\epsilon \quad\text{if}\quad \mathrm{Im}k\geq\tfrac{1}{2}\mu-\epsilon,$$

then

$$|g_n(k,r)|\leq(McV_0/\mu)^n/n!$$

uniformly in k and r. Hence the series converges absolutely and uniformly. Similarly one establishes the existence of the derivative and thus the analyticity of $f_0(k,r)$ and $f_0(k)$ everywhere, except on the cut.

If we define functions $h_n(\alpha,k,r)$ by the recursion

$$h_0(\alpha,k,r)=e^{-\alpha r}$$

$$h_n(\alpha,k,r)=V_0\int_{\alpha+\mu}^{\infty}\frac{d\alpha'}{\alpha'}\frac{\rho(\alpha'-\alpha)}{\alpha'+2ik}h_{n-1}(\alpha',k,r),\quad n\geq 1,$$

then

$$g_n(k,r)=h_n(0,k,r),$$

and

$$h(\alpha,k,r)=\sum_{n=0}^{\infty}h_n(\alpha,k,r)$$

converges absolutely and uniformly so long as k stays at least a fixed distance away from the cut which runs from $k=\frac{1}{2}i(\alpha+\mu)$ upwards. The function $h(\alpha,k,r)$ satisfies the integral equation

$$h(\alpha,k,r)=e^{-\alpha r}+V_0\int_{\alpha+\mu}^{\infty}\frac{d\alpha'}{\alpha'}\cdot\frac{\rho(\alpha'-\alpha)}{\alpha'+2ik}h(\alpha',k,r), \tag{10.9}$$

which determines $h(\alpha,k,r)$ explicitly in terms of $h(\alpha',k,r)$ for $\alpha'\geq\alpha+\mu$. For

$$k=\tfrac{1}{2}i(\alpha+\mu)-\tfrac{1}{2}i\epsilon,$$

we find

$$h(\alpha,k,r)=\frac{V_0\rho(\mu)}{\alpha+\mu}h(\alpha+\mu,\,k,r)\log\left(\frac{\alpha+\mu}{\epsilon}\right)+O(1). \tag{10.10}$$

Since

$$f_0(k)=1+V_0\int_{\mu}^{\infty}\frac{d\alpha}{\alpha}\frac{\rho(\alpha)}{\alpha+2ik}h(\alpha,k,0),$$

we have

$$\lim_{|k|\to\infty}f_0(k)=1$$

everywhere in the complex plane. Consequently, the S matrix is an analytic function of k, regular in the complex plane cut along the positive imaginary axis from $k=\frac{1}{2}i\mu$ to infinity, continuous on the cut, except near the point $k=\frac{1}{2}i\mu$, where it is $O[\log(2ik+\mu)]$. Furthermore,

$$\lim_{|k|\to\infty}S_0(k)=1.$$

One may then use Cauchy's theorem to express the real

[70] R. E. Peierls, Proc. Roy. Soc. (London) **A253**, 16 (1959).

[71] This is a very strong assumption, since it implies not only that (3.14) holds for any $a<\frac{1}{2}\mu$, but also that $V(r)$ is an analytic function of r regular in the open right half of the complex plane.

[72] For explicit extension to $l>0$, see A. Martin, Nuovo cimento **15**, 99 (1960); and D. I. Fivel and A. Klein, preprint.

part of S_0 (for real k) in terms of its imaginary part, bound state contributions, and an integral over the cut along the positive imaginary axis.[15] Because of the last contribution such a dispersion relation is not very useful.

(f) Generalized Bargmann Potentials

Suppose we are given an arbitrary potential $V^{(0)}(r)$ and the corresponding functions $\varphi_l^{(0)}(k,r)$, $f_l^{(0)}(k,r)$ and $f_l^{(0)}(k)$. This potential need not satisfy (3.1), but we take it so that, except for isolated singularities $f_l^{(0)}(k)$ possesses an analytic extension into the upper half plane. In fact $V^{(0)}(r)$ may even be the Coulomb potential; in that case (3.4) is replaced by[73]

$$\lim_{r\to\infty} \exp i[kr-\eta k^{-1}\log r]f_0^{(0)}(k,r)=i^l.$$

We want to write[74] down the potential $\Delta V(r)$ which, if added to $V^{(0)}(r)$, causes a new $S_l(k)$ that differs from the old by a finite number of poles and zeros

$$S_l(k)=S_l^{(0)}(k)R(k)/R(-k), \tag{10.11}$$

where $R(k)$ is a rational function with N simple poles at $k=\beta_n(\mathrm{Im}\beta_n>0)$ and N simple zeros at $k=\alpha_n$, and which tends to one at infinity[75]:

$$R(k)=\Pi[(k-\alpha)/(k-\beta)]. \tag{10.12}$$

Among the α's we distinguish between those in the ***upper*** half-plane, which we call γ, $\mathrm{Im}\gamma>0$, and those in the *lower*, which we call κ, $\mathrm{Im}\kappa<0$.

We now form the functions

$$x_\beta(k,r)\equiv(\beta^2-k^2)^{-1}W[\varphi_l^{(0)}(\beta,r),f_l^{(0)}(k,r)], \tag{10.13}$$

$$y_\beta(k,r)\equiv(\beta^2-k^2)^{-1}W[\varphi_l^{(0)}(\beta,r),\varphi_l^{(0)}(k,r)], \tag{10.14}$$

$$\begin{cases} x_{\gamma\beta}(r)\equiv x_\beta(-\gamma,r) \\ x_{\kappa\beta}(r)\equiv x_\beta(\kappa,r)-i^lC_\kappa y_\beta(\kappa,r), \end{cases} \tag{10.15}$$

where C_κ are a set of arbitrary real constants.

Notice that we can also write

$$x_\beta(k,r)=\int_0^r dr'\,\varphi_l^{(0)}(\beta,r')f_l^{(0)}(k,r') +(k^2-\beta^2)^{-1}f_l^{(0)}(k), \tag{10.16}$$

$$y_\beta(k,r)=\int_0^r dr'\,\varphi_l^{(0)}(\beta,r')\varphi_l^{(0)}(k,r'). \tag{10.17}$$

We then define N functions $K_\beta(r)$ by the N equations[76]

$$\begin{aligned}\textstyle\sum_\beta x_{\gamma\beta}(r)K_\beta(r)&=-f_l^{(0)}(-\gamma,r),\\ \textstyle\sum_\beta x_{\kappa\beta}(r)K_\beta(r)&=-f_l^{(0)}(\kappa,r)+i^lC_\kappa\varphi_l^{(0)}(\kappa,r).\end{aligned} \tag{10.18}$$

The claim is that when

$$\Delta V(r)\equiv 2\frac{d}{dr}\sum_\beta K_\beta(r)\varphi_l^{(0)}(\beta,r) \tag{10.19}$$

is added to $V^{(0)}(r)$, it produces the $S_l(k)$ of (10.11) and, furthermore, there are bound states of energies $-\hbar^2\kappa^2/2\mu$ in addition to those of $V(r)$.

A few steps of simple algebra show that the functions

$$h(k,r)\equiv f_l^{(0)}(k,r)+\textstyle\sum_\beta K_\beta(r)x_\beta(k,r), \tag{10.20}$$

$$g(k,r)\equiv \varphi_l^{(0)}(k,r)+\textstyle\sum_\beta K_\beta(r)y_\beta(k,r), \tag{10.21}$$

satisfy the differential equations

$$-h''+[l(l+1)r^{-2}+V^{(0)}+\Delta V-k^2]h =\textstyle\sum_\beta \rho_\beta(r)x_\beta(k,r), \tag{10.22}$$

$$-g''+[l(l+1)r^{-2}+V^{(0)}+\Delta V-k^2]g =\textstyle\sum_\beta \rho_\beta(r)y_\beta(k,r), \tag{10.23}$$

where

$$\rho_\beta(r)\equiv -K_\beta''+[l(l+1)r^{-2}+V^{(0)}+\Delta V-\beta^2]K_\beta. \tag{10.24}$$

Now by the definitions (10.15) and (10.18) we have

$$h(-\gamma,r)\equiv 0,\quad h(\kappa,r)\equiv i^lC_\kappa g(\kappa,r), \tag{10.25}$$

insertion of which in (10.22) implies by (10.23) that

$$\textstyle\sum_\beta x_{\alpha\beta}(r)\rho_\beta(r)=0$$

for all α. We may conclude that[76] $\rho_\beta(r)=0$ for all β. The functions h and g thus both satisfy the Schrödinger equation with the new potential $V=V^{(0)}+\Delta V$.

Next we look at the boundary values. As $r\to\infty$ it is readily seen that

$$x_\beta(k,r)\sim -\tfrac{1}{2}(-)^l(k+\beta)^{-1}f_l^{(0)}(-\beta)e^{-i(k+\beta)r},$$
$$y_\beta(\kappa,r)\sim \tfrac{1}{4}i\kappa^{-l-1}(\beta-\kappa)^{-1}f_l^{(0)}(\kappa)f_l^{(0)}(-\beta)e^{-i(\beta-\kappa)r}.$$

The equations for $K_\beta(r)$, (10.18), thus become for large r

$$\textstyle\sum_\beta(\alpha-\beta)^{-1}f_l^{(0)}(-\beta)e^{-i\beta r}K_\beta(r)=-2i^{-l},$$

from which it follows that there exists a set of N constants a_β such that

$$\lim_{r\to\infty} f_l^{(0)}(-\beta)e^{-i\beta r}K_\beta(r)=-2i^{-l}a_\beta \tag{10.26}$$

and

$$1-\textstyle\sum_\beta(\alpha-\beta)^{-1}a_\beta=0. \tag{10.27}$$

We may immediately infer that

$$R(k)=1-\textstyle\sum_\beta(k-\beta)^{-1}a_\beta, \tag{10.28}$$

[73] $\eta=\mu cZZ'\alpha/\hbar$, where μ is the reduced mass, Z and Z' are the two charges in units of the electronic charge, α is the fine structure constant "1/137," and c is the velocity of light.

[74] The treatment below is a generalization of that of W. R. Theis, Z. Naturforsch. **11a**, 889 (1956), to include bound states. One may obtain these potentials also by solving the Gel'fand Levitan equation, a procedure due to Bargmann, unpublished.

[75] We use a simplified notation such as Σ_β to indicate a sum over the β_n from 1 to N.

[76] It is clear that $\det[x_{\alpha\beta}(r)]\neq 0$.

both sides being rational functions of k with the same zeros and poles and the same limit as $|k|\to\infty$.

The asymptotic behavior of the function $h(k,r)$ of (10.20) for large r is now easily seen to be

$$h(k,r)\sim i^l e^{-ikr}R(-k),$$

which proves that

$$f_l(k,r)=h(k,r)/R(-k). \tag{10.29}$$

The function $g(k,r)$ of (10.21) is a regular solution of (2.10). From (10.21), (4.1), and (10.29) we see that

$$g(k,r)=\tfrac{1}{2}ik^{-l-1}[f_l^{(0)}(-k)R(-k)f_l(k,r) \\ -(-)^l f_l^{(0)}(k)R(k)f_l(-k,r)]. \tag{10.30}$$

It follows that $S_l(k)$ is indeed given by (10.11). Moreover, by (10.25) and (10.29),

$$f_l(\kappa,r)=i^lC_\kappa\prod\frac{\kappa+\beta}{\kappa+\alpha}g(\kappa,r), \tag{10.31}$$

both sides being regular at $r=0$ and decreasing exponentially at infinity; κ is thus indeed a bound state. Since for given zeros of $f_l(k)$ in the lower half-plane (4.5) and (4.16) define $f_l(k)$ uniquely [see (5.21′)], we may conclude from (10.11) that

$$f_l(k)=f_l^{(0)}(k)R(k), \tag{10.32}$$

and hence from (10.30) and (4.1)

$$\varphi_l(k,r)=g(k,r). \tag{10.33}$$

We can also evaluate the normalization integral of $\varphi_l(\kappa,r)$. If we use the equations between (7.3) and (7.4). we get

$$N_\kappa^2=\int_0^\infty dr|\varphi_l(\kappa,r)|^2 \\ =(i\kappa)^{-l}\frac{\prod_{\alpha\neq\kappa}(\alpha^2+|\kappa|^2)}{\prod_\beta(\beta^2+|\kappa|^2)}\cdot\frac{f_l^{(0)}(\kappa)}{C_\kappa}. \tag{10.34}$$

The potential ΔV can be written in a somewhat simpler form. If we solve the set of Eqs. (10.18),

$$K_\beta(r)=-\sum_\alpha[x^{-1}(r)]_{\beta\alpha}U_\alpha^{(l)}(r), \tag{10.35}$$

where

$$U_\gamma^{(l)}(r)\equiv f_l^{(0)}(-\gamma,r) \\ U_\kappa^{(l)}(r)\equiv f_l^{(0)}(\kappa,r)-i^lC_\kappa\varphi_l^{(0)}(\kappa,r), \tag{10.36}$$

then we can write (10.19),

$$\Delta V(r)=-2(d^2/dr^2)\log\det[x_{\alpha\beta}(r)], \tag{10.37}$$

since it follows from (10.16) and (10.17) that

$$U_\alpha^{(l)}(r)\varphi_l^{(0)}(\beta,r)=(d/dr)x_{\alpha\beta}(r).$$

To summarize then, the potential $V=V^{(0)}+\Delta V$, where ΔV is given by (10.37), produces the functions $\varphi_l(k,r)$ and $f_l(k,r)$ given by (10.33) and (10.29), bound states of energy $\hbar^2\kappa^2/2\mu$ with wave functions $\varphi_l(\kappa,r)$ whose normalization is given by (10.34), and the S matrix element (10.11), or

$$S_l(k)=S_l^{(0)}(k)\prod\frac{k-\gamma}{k+\gamma}\cdot\frac{k+\beta}{k-\beta}\cdot\frac{k-\kappa}{k+\kappa}. \tag{10.11′}$$

We are free to choose a γ equal to a $-\kappa$. In that case S_l contains no pole and no zero because of the bound state. The potential ΔV is real if we choose the γ's and β's either purely imaginary or else in pairs symmetric with respect to the imaginary axis, and the κ's purely imaginary. The S_l of (10.11) is then unitary (if $S_l^{(0)}$ is). But one may also relax these requirements and make ΔV complex as an "optical" potential in order to simulate absorption.

If we choose $V^{(0)}\equiv0$ then we get the Bargmann potentials,[77] which lead to a rational $S_l(k)$. They are often very useful for the construction of simple models. A potential which leads to a rational $S_l(k)$ for one $l=l_0$ will in general not lead to a rational $S_l(k)$ for $l\neq l_0$. Since for $l=0$ the functions that enter in $V(r)$ are all exponentials (multiplied by sines and cosines if we choose complex β's and γ's), the Bargmann potentials for the S-wave have in general exponential tails.[78] This shows that an exponential asymptotic form of the potential does not *necessarily* lead to infinitely many poles of S_l in the upper half-plane, although it does in general.[79]

If $l\neq0$ then the functions entering the Bargmann potentials are spherical Bessel functions and thus they contain inverse powers of r. As a result, they generally have asymptotic tails r^{-n}, where $n\geq3$. It has been shown[80] that a sufficient condition for a Bargmann potential to have an exponential tail is that $f_l(k)=f_l(0)+O(k^{2l})$ as $k\to0$.

We may look at some special cases. If we take one $\gamma=ia$, one $\beta=ib$, $V^{(0)}=0$, and $l=0$,[77]

$$f_0(k)=(k-ia)/(k-ib),$$

or

$$k\cot\delta_0=[ab/(b-a)]+[k^2/(b-a)],\ b\geq0,\ a\geq0, \tag{10.38}$$

then the effective range approximation is exact. The potential that produces this phaseshift is

$$V(r)=-\frac{8b^2}{b^2-a^2}\left[\frac{e^{br}}{b-a}+\frac{e^{-br}}{b+a}\right]^{-2}. \tag{10.39}$$

If we set $a=0$ then we get a zero-energy resonance $(f_0(0)=0)$

$$\tan\delta_0=b/k;$$

the potential that produces it is

$$V(r)=-2b^2\operatorname{sech}^2 br.$$

[77] V. Bargmann, Revs. Modern Phys. 21, 488 (1949).
[78] In special cases they may not, as will be seen below.
[79] See end of Sec. 10(d) and Peierls.[70]
[80] By T. Fulton (unpublished) and Newton.[55]

On the other hand, if we set $b=0$ then the phaseshift becomes

$$\tan\delta_0=-a/k.$$

Since then $\delta_0(0)-\delta_0(\infty)=-\frac{1}{2}\pi$, the Levinson theorem (5.15) is violated. The potential that produces this phaseshift is

$$V(r)=2a^2(1+ar)^{-2}.$$

Notice that in this case, which violates (3.1), the Jost function has a pole at $k=0$.

We can also make the effective range approximation exact with a bound state. A case of interest is the deuteron. The phaseshift has the form (10.38) with a replaced by κ, the binding energy being $\hbar^2\kappa^2/2\mu$. The potentials that produce that phaseshift and bound state are[81]

$$V_c(r)=-4\kappa\frac{d}{dr}\left\{\sinh br\frac{g_c(\kappa,r)}{g_c(\kappa+b,r)-g_c(\kappa-b,r)}\right\}, \qquad (10.40)$$

where

$$g_c(k,r)=k^{-1}[e^{-kr}+c\sinh kr].$$

The normalized bound state wave function is

$$\phi(r)=2\left(\frac{c\kappa}{b^2-\kappa^2}\right)^{\frac{1}{2}}\frac{\sinh br}{g_c(\kappa+b,r)-g_c(\kappa-b,r)}. \qquad (10.41)$$

The potentials (10.40) have asymptotic tails proportional to $e^{-2\kappa r}$, except when $c=-4$, in which case it decreases more rapidly.[82]

[81] R. G. Newton, Phys. Rev. **105**, 763 (1957).

[82] It is a general property of the potentials producing a given phaseshift and given bound states of smallest binding energy $\hbar^2K_1^2/2\mu$ and largest binding energy $\hbar^2K_2^2/2\mu$ that, if one of them decreases asymptotically more rapidly than $\exp(-2K_2r)$ then it is the only one with that property, and if one of them decreases less rapidly than $\exp(-2K_1r)$ then they all do; cf. Newton.[55]

An amusing case is the one for which $S_0(k)\equiv 1$, i.e., which causes no s wave scattering whatever, at any energy, but which causes a bound state of zero energy. The potentials that do that are[77,83]

$$V(r)=-6(d/dr)[r^2/(c^2+r^3)].$$

The normalized bound state wave function[84] is

$$\phi(r)=\sqrt{3}cr/(c^2+r^3),$$

while

$$f_0(k,r)=e^{-ikr}-[3rk^{-2}/(c^2+r^3)][ikre^{-ikr}+e^{-ikr}-1].$$

The Levinson theorem (5.15) is again not fulfilled. One can similarly find the potential for which

$$f_0(k)=1+\kappa^2/k^2,$$

and which therefore has a bound state of binding energy $\hbar^2\kappa^2/2\mu$, but which causes no s scattering. This potential, however, has infinitely many singularities on the real axis.[83]

We can also use the preceding procedure to write down the potentials with a hard core or a Coulomb contribution but whose S matrix differs, for one l value, from that for a *pure* hard core or pure Coulomb field by a rational factor. The construction of such examples is left as an exercise to the reader.

The Bargmann potentials have been generalized by Fulton and Newton[85] to the case with coupling between two angular momenta. The resulting potentials constitute the only tensor forces for which the Schrödinger equation is known to have a solution in closed form. They have been applied to the case of low-energy neutron-proton scattering.[86]

[83] H. E. Moses and S. F. Tuan, Nuovo cimento **13**, 197 (1959).

[84] This shows that when (3.1) is violated then there can be a bound state of $l=0$ with zero binding energy; when (3.1) is satisfied, that is impossible.

[85] T. Fulton and R. G. Newton, Nuovo cimento **3**, 677 (1956).

[86] R. G. Newton and T. Fulton, Phys. Rev. **107**, 1103 (1957).